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The Global Renormalization Group Trajectory in a Critical Supersymmetric Field Theory on the Lattice $\mathbb{Z}^3$

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Abstract We consider an Euclidean supersymmetric field theory in $\mathbb{Z}^3$ given by a supersymmetric Φ^4 perturbation of an underlying massless Gaussian measure on scalar bosonic and Grassmann fields with covariance the Green's function of a (stable) Lévy random walk in $\mathbb{Z}^3$. The Green's function depends on the Lévy-Khintchine parameter $\alpha = \frac{3+\epsilon}{2}$ with $0 < \alpha < 2$. For $\alpha = \frac{3}{2}$ the Φ^4 interaction is marginal. We prove for $\alpha - \frac{3}{2} = \frac{\epsilon}{2} > 0$ sufficiently small and initial parameters held in an appropriate domain the existence of a global renormalization group trajectory uniformly bounded on all renormalization group scales and therefore on lattices which become arbitrarily fine. At the same time we establish the existence of the critical (stable) manifold. The interactions are uniformly bounded away from zero on all scales and therefore we are constructing a non-Gaussian supersymmetric field theory on all scales. The interest of this theory comes from the easily established fact that the Green's function of a (weakly) self-avoiding Lévy walk in $\mathbb{Z}^3$ is a second moment (two point correlation function) of the supersymmetric measure governing this model. The rigorous control of the critical renormalization group trajectory is a preparation for the study of the critical exponents of the (weakly) self-avoiding Lévy walk in $\mathbb{Z}^3$.

Keywords Lattice renormalization group · Supersymmetry · Self-avoiding Levy processes

1 Introduction

It was observed long ago [25, 26], that the Green's function of weakly self avoiding simple random walks (SAW) on a lattice $\mathbb{Z}^d$ can be expressed as a correlation function in a

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supersymmetric field theory. This can be shown rigorously by the same derivation as in [9–11] for SAWs on hierarchical lattices. Consider instead of simple random walks the more general case of continuous time (stable) Lévy walks whose scaling limits are stable Lévy distributions [16, 24]. Such walks can be realized as jump processes with probability distributions permitting long range jumps [16]. Their characteristic functions are given by the Lévy-Khintchine formula with characteristic exponent α, $0<\alpha\le 2$ [16], $\alpha=2$ corresponding to simple random walks. The Green's function of continuous time weakly self avoiding Lévy walks (SALW) can also be realized as a two point correlation function in a supersymmetric field theory by the same derivation as in [9–11]. This paper is concerned with proving the existence of a critical uniformly bounded renormalization group (RG) trajectory for the interactions in the underlying supersymmetric field theory corresponding to the class of SALWs where $\alpha=\frac{3+\varepsilon}{2}$ with $0<\varepsilon<1$ and ε held small. The case $\alpha=\frac{3}{2}$ corresponds to mean field theory. Uniformity is with respect to the lattice scale which changes with each step of the renormalization group map. We find that the interactions are non-vanishing at all renormalization group scales, which is the lattice version of a non-Gaussian fixed point. This gives the foundation for the study of the Green's function of SALWs in the scaling limit which is postponed to the sequel. Ultimately one would like to be able to extract from this the end-to-end distance behaviour for SALWs.

The supersymmetric field theory in question is a lattice supersymmetric generalization of the model considered in [13]. We describe it informally here and leave the details for the next section. Let Δ be the standard Laplacian in $\mathbb{Z}^3$. Then for $x,\ y\in\mathbb{Z}^3$, and $0<\alpha<2$, $C(x-y)=(-\Delta)^{-\alpha/2}(x-y)$ is the Green's function of a stable Lévy walk. Let $\varphi_1,\ \varphi_2$ be independent identically distributed Gaussian random fields in $\mathbb{Z}^3$ with covariance $\frac{1}{2}C$. Let $\varphi=\varphi_1+i\varphi_2$ and $\bar\varphi$ its complex conjugate. Introduce a pair of Grassmann fields $\psi,\bar\psi$ of degree 1 and -1 respectively. Let $\Phi=(\varphi,\psi)$ and $\bar\Phi=(\bar\varphi,\bar\psi)$. The inner product is given by $(\Phi,\Phi)=\Phi\bar\Phi=\varphi\bar\varphi+\psi\bar\psi$. Let $\Lambda\subset\mathbb{Z}^3$ be a finite subset. Define

$$V_0(\Lambda,\Phi)=g_0\int_\Lambda dx(\Phi\bar\Phi)^2(x)+\tilde\mu_0\int_\Lambda dx\Phi\bar\Phi(x) \tag{1.1}$$

where the coupling constant $g_0>0$ and dx is the counting measure in $\mathbb{Z}^3$. Then our model in finite volume Λ is defined by the supermeasure

$$d\mu_\Lambda(\Phi)=d\mu_{C_\Lambda}(\Phi)e^{-V_0(\Lambda,\Phi)} \tag{1.2}$$

where C_Λ is the restriction of C to the points of Λ and $d\mu_{C_\Lambda}(\Phi)$ is the Gaussian supermeasure

$$d\mu_{C_\Lambda}(\Phi)=\prod_{x\in\Lambda}d\varphi_1(x)d\varphi_2(x)d\psi(x)d\bar\psi(x)\,e^{-(\Phi,C_\Lambda^{-1}\bar\Phi)_{L^2(\Lambda)}} \tag{1.3}$$

Integration over the Grassmann fields is Berezin integration and $d\mu_\Lambda(\Phi)$ is interpreted as a linear functional on the Grassman algebra (generated by the $\psi,\bar\psi$ with coefficients which are functionals of the $\varphi,\bar\varphi$). An important fact is that the potential $V_0(\Lambda,\Phi)$ is supersymmetric (supersymmetry in this context and some of its consequences are given in the Sect. 1.1). As a consequence we have that the supermeasure $d\mu_\Lambda(\Phi)$ is normalized:

$$\int d\mu_\Lambda(\Phi)\ 1=1 \tag{1.4}$$

The parameters of the supermeasure μ_Λ defined in (1.2) correspond to those of SALWs. Thus g_0 measures the strength of self-repulsion and $\tilde\mu_0$ the killing rate of a weakly self-avoiding Lévy walk. The reader will get a full dictionary in [9–11] where the end-to-end distance behaviour was studied for SAWs in a four dimensional hierarchical lattice with the help of supersymmetry.

We give an informal description of the results of this paper. We will choose $\alpha = \frac{3+\varepsilon}{2}$ with $0 < \varepsilon < 1$, in particular we hold $\varepsilon > 0$ very small. We will take Λ to be a very large cube. By successive RG transformations we will get a sequence of measures (the RG trajectory of measures) living in smaller and smaller cubes in finer and finer lattices till we arrive at a fixed small cube in a very fine lattice. This will take $\log \Lambda$ steps. At every step the measure is a new Gaussian measure times a new supersymmetric density. The Gaussian measure is characterized by a covariance and the sequence of covariances converge to a smooth continuum covariance. The supersymmetric density incorporates the interactions. The principal information is in the local interactions incorporated in local potentials of the above type albeit with new parameters (coupling constants) and on a finer lattice. The other interactions are contracting (irrelevant) in an appropriate sense and are expressed in the form of polymer activities. The coupling constants and polymer activities give coordinates of the RG trajectory. These coordinates provide Banach spaces of interactions which permit a rigorous study of the Wilson RG [28] avoiding real space renormalization group pathologies, [21, 22], related to the Griffiths singularity problem in disordered systems [20]. See [5] for a review of these pathologies. The goal of this paper is to study the RG trajectory of these coordinates in the infinite volume limit which makes sense for these coordinates. The true infinite volume limit and the scaling limit will be taken at the level of correlation functions.

In Sect. 1 we define the model, introduce supersymmetry and develop some of its consequences. The RG analysis of this paper is based on the finite range multiscale expansion of covariances of [8]. We summarize the basic results of [8] pertinent to this paper in Theorem 1.1. This is an alternative to the Kadanoff- Wilson block spin RG developed extensively by Gawedzki and Kupiainen [17–19], and Balaban [2–4]. A crucial simplification arises due to the finite range of the fluctuation covariances: Cluster expansions are no longer needed in the control of the fluctuation integration which is an essential step of RG transformations. As a result all estimates are local in character. In this section we also define lattice polymers and polymer activities.

In Sect. 2 we introduce norms which will measure the size of polymer activities. These norms are suggested by those in the continuum analysis of [13][1] but now take account of the presence of Grassmann fields. The choice of these norms was inspired by discussions with David Brydges. They are closely related to norms which will appear in the forthcoming study of self-avoiding simple random walks in four dimensions by Brydges and Slade [14].

In Sect. 3 we define the RG map as we will use it and in Sect. 4 apply it to our model. In particular we develop second order perturbation theory. The task is to control the contributions from the remainder and this is taken up in the next section.

Section 5 gives the basic estimates that we will need for the control of the RG trajectory. These estimates are extensions of those in Sect. 5 of [13]. The latter paper studied a critical bosonic theory in the continuum. Our present estimates take account of the presence of Grassmann variables as well as the lattice which has led to a considerable number of new details. The upshot is Theorem 5.1.

Section 6 is devoted to the proof of existence of the stable manifold: there exists an initial critical mass $\tilde{\mu}_0$ which is a Lipshitz continuous function of the coupling constant g_0

[1] A. Abdesselam in [1] corrected an error which occurs in [7, 13] where it is wrongly asserted that certain normed spaces are complete. This problem was resolved in [1] by some changes in definitions which fortunately are such that, as noted in [1], the estimates, theorems and proofs of [7, 13] remain true without change. On the lattice however the function space subtleties encountered in [1] disappear.

such that RG trajectory is bounded uniformly on all scales. The proof is established by a combination of three theorems, namely Theorems 6.1, 6.2, and 6.3.

Finally we observe that the coupling constant g_n is uniformly bounded away from 0 at all scales $n \geq 0$. As a result the global RG trajectory gives rise to a non-Gaussian field theory. We remark that in a continuum version of this model with a cutoff modelled on that of [13] one can prove more: the continuum RG trajectory ends at a non-trivial fixed point. But the notion of a fixed point is devoid of meaning for lattice field theories because the RG map even in infinite volume does not give an autonomous action on a fixed Banach space.

1.1 Definitions, Model, Supersymmetry

Let e_1, e_2, e_3 be the standard basis of unit vectors specifying the orientation of $\mathbb{Z}^3$. We let ∂_μ denote the forward lattice derivative in direction e_μ and ∂_μ^* its $L^2(\mathbb{Z}^3)$ adjoint. The latter is the backward derivative. Then the lattice Laplacian Δ in $\mathbb{Z}^3$ is defined by

$$-\Delta = \sum_{\mu=1}^{3} \partial_\mu^* \partial_\mu \tag{1.5}$$

Let $\hat{\Delta}(p)$ be the Fourier transform of the integral kernel of Δ in $\mathbb{Z}^3$, namely

$$\hat{\Delta}(p) = 2\sum_{\mu=1}^{3} (\cos(p_\mu) - 1) \tag{1.6}$$

Let $\alpha = \frac{3+\varepsilon}{2}$ be a real number with $0 < \alpha < 2$. Then the Green's function of a (stable) Lévy walk in $\mathbb{Z}^3$ is given by

$$C(x-y) = (-\Delta)^{-\alpha/2}(x-y) = \int_{[-\pi,\pi]^3} \frac{d^3p}{(2\pi)^3}\, e^{ip(x-y)}\,(-\hat{\Delta}(p))^{-\alpha/2} \tag{1.7}$$

C is positive-definite and therefore qualifies as the covariance of a Gaussian random field in $\mathbb{Z}^3$. We introduce a pair of independent identically distributed Gaussian random fields $\varphi_1,\ \varphi_2$ with mean 0 and covariance

$$E(\varphi_j(x)\varphi_j(y)) = \frac{1}{2}C(x-y) \tag{1.8}$$

for $j = 1, 2$.

Let $\varphi(x) = \varphi_1(x) + i\varphi_2(x)$ be a complex scalar field and $\bar{\varphi}(x)$ its complex conjugate. On the space of functionals of $\varphi, \bar{\varphi}$ we have the Gaussian probability measure

$$d\mu_C(\phi) = d\mu_{\frac{1}{2}C}(\phi_1)\, d\mu_{\frac{1}{2}C}(\phi_2) \tag{1.9}$$

Then each of $\varphi, \bar{\varphi}$ has zero covariance and

$$E(\bar{\varphi}(x)\varphi(y)) = C(x-y) \tag{1.10}$$

Grassmann Algebra and Integration

Let $\Lambda \subset \mathbb{Z}^3$ be a bounded subset. $\mathcal{F}_\Lambda$ represents the algebra of $\mathbb{C}$ valued functionals of the fields $\{\phi, \bar{\phi} : \Lambda \to \mathbb{C}\}$. Let $\psi(x), \bar{\psi}(x)$ for all $x \in \Lambda$ be the (anti-commuting) Grassman elements of degree $1, -1$ respectively. Following standard usage we will refer to them as (scalar) *fermions*. We denote by Ω_Λ the Grassman algebra generated by the $\psi(x), \bar{\psi}(y)$ by multiplication and linear sums for all $x, y \in \Lambda$ with coefficients in $\mathcal{F}_\Lambda$. The Grassmann algebra is naturally graded $\Omega_\Lambda = \oplus_p \Omega_\Lambda^p$ where the integer p is the degree and each Ω_Λ^p is a $\mathcal{F}_\Lambda$ module. Ω_Λ^0 is an algebra. Because of the anticommuting property of the generators, and because Λ is a finite lattice an element of Ω_Λ^p is a finite sum of degree p elements with coefficients in $\mathcal{F}_\Lambda$. For example, an element F_Λ of Ω_Λ^0 can be uniquely represented as

$$F_\Lambda(\varphi, \psi) = \sum_{p \geq 0} \int_{\Lambda^{2p}} \prod_{j=1}^{p} dx_j dy_j \; F_{\Lambda,2p}(\varphi; x_1, \ldots, x_p, y_1, \ldots, y_p) \prod_{j=1}^{p} \psi(x_j)\bar{\psi}(y_j) \quad (1.11)$$

where dx is the counting measure in $\mathbb{Z}^3$. The coefficients, $F_{\Lambda,2p}(\varphi; x_1, \ldots, x_p, y_1, \ldots, y_p)$, are antisymmetric in $(x_1, \ldots, x_p)$ and in $(y_1, \ldots, y_p)$. When Λ is a finite subset the above multiple sum is finite. In the following we will often refer to the coefficients $F_{\Lambda,2k}$ above as *bosonic coefficients*. Here and in the following we suppress indicating the dependence of the bosonic coefficients on $\bar{\varphi}$.

These considerations are of course valid for a lattice $(\delta\mathbb{Z})^3$ for any lattice spacing δ with the corresponding notations $\Lambda_\delta, \mathcal{F}_{\Lambda_\delta}, \Omega_{\Lambda_\delta}, dx$ being δ^3 times the counting measure in $(\delta\mathbb{Z})^3$.

Now we define fermionic expectation (integration) using Berezin integration which we review briefly and set up our conventions. Berezin integration is a linear map $\Omega_\Lambda \to \Omega_\Lambda$ which satisfies

$$\int d\psi(x) F_\Lambda(\psi, \bar{\psi}, \phi, \bar{\phi}) = \pi^{-1/2} \frac{\partial}{\partial \psi(x)} F_\Lambda \quad (1.12)$$

where $F_\Lambda \in \Omega_\Lambda$ and the fermionic derivative $\frac{\partial}{\partial \psi(x)}$ is an antiderivation: If $f \in \Omega_\Lambda^p$ and $g \in \Omega_\Lambda^q$ then

$$\frac{\partial}{\partial \psi(x)}(fg) = \frac{\partial}{\partial \psi} fg + (-1)^p f \frac{\partial}{\partial \psi(x)} g$$

Integration with respect to $d\bar{\psi}(x)$ is given by the same formula with $\frac{\partial}{\partial \bar{\psi}(x)}$ on the right hand side. Multiple integration is repeated integration using the above rule, keeping in mind that fermionic derivatives anticommute.

Define $C_\Lambda(x - y) = C(x - y) : x, y \in \Lambda$. We consider this as a $|\Lambda| \times |\Lambda|$ dimensional positive definite symmetric matrix with x, y labelling the entries. Then we define the fermionic expectation $E_{f,\Lambda}$ as a linear map $\Omega_\Lambda \to \mathcal{F}_\Lambda$ as follows: Let $F_\Lambda \in \Omega_\Lambda$. We adopt the convention $F_\Lambda(\psi, \phi) \equiv F_\Lambda(\psi, \bar{\psi}, \phi, \bar{\phi})$. Then

$$E_{f,\Lambda}(F_\Lambda) = \int d\mu_{C_\Lambda}(\psi) F_\Lambda(\psi, \phi) \quad (1.13)$$

where

$$\int d\mu_{C_\Lambda}(\psi) F_\Lambda(\psi, \phi)) = (\det \pi C_\Lambda)^{|\Lambda|} \int \prod_{x \in \Lambda} (d\psi(x) d\bar{\psi}(x)) \, e^{-(\psi, C_\Lambda^{-1} \bar{\psi})_{L^2(\Lambda)}} F_\Lambda(\psi, \phi) \quad (1.14)$$

We call $d\mu_{C_\Lambda}(\psi)$ a fermionic Gaussian measure and use the terminology measure and expectation interchangeably. It is not difficult to show that we have a fermionic counterpart of the bosonic Gaussian formula, namely

$$\int d\mu_{C_\Lambda}(\psi)F_\Lambda(\psi,\phi) = e^{\int_{\Lambda\times\Lambda} dxdy C_\Lambda(x-y)\frac{\partial}{\partial\psi(x)}\frac{\partial}{\partial\bar{\psi}(y)}} F_\Lambda(\psi,\phi)|_{\psi=\bar{\psi}=0} \tag{1.15}$$

where dx is the counting measure. The fermionic expectation above annihilates the component of $F_\Lambda \notin \Omega^0_\Lambda$.

Note that the expectation of a product of two ψ or of two $\bar{\psi}$ vanishes whereas if $x, y \in \Lambda$

$$E_{f,\Lambda}(\bar{\psi}(x)\psi(y)) = C(x-y) \tag{1.16}$$

More generally, if $x_j, y_j \in \Lambda,\ j = 1, 2, \ldots, n$

$$E_{f,\Lambda}\left(\prod_{j=1}^{n} \bar{\psi}(x_j)\psi(y_j)\right) = \det\left(C(x_j - y_k)\right)^n_{j,k=1} \tag{1.17}$$

We define the field $\Phi(x)$ (called superfield in anticipation) as the pair

$$\Phi(x) = (\varphi(x), \psi(x)) \tag{1.18}$$

with the scalar product

$$(\Phi(x), \Phi(y)) = \Phi(x)\bar{\Phi}(y) = \varphi(x)\bar{\varphi}(y) + \psi(x)\bar{\psi}(y) \tag{1.19}$$

More generally if $A(x, y)$ is a matrix for $x, y \in \Lambda$ we define

$$\begin{aligned}(\Phi, A\Phi)_{L^2(\Lambda)} &= \int_{\Lambda\times\Lambda} dxdy\ \Phi(x)A(x,y)\bar{\Phi}(y)\\ &= \int_{\Lambda\times\Lambda} dxdy\ (\varphi(x)A(x,y)\bar{\varphi}(y) + \psi(x)A(x,y)\bar{\psi}(y))\end{aligned} \tag{1.20}$$

Let $F_\Lambda(\Phi)$ belong to Ω_Λ. F_Λ also depends on $\bar{\Phi}$ but here and in the following this is not explicitly indicated. Since $F_\Lambda(\Phi) \in \Omega_\Lambda$ it has the representation (1.11). We define the expectation E_Λ as a linear map $\Omega_\Lambda \to \mathbf{C}$ obtained by combining the bosonic and fermionic expectations: If $F_\Lambda \in \Omega_\Lambda$ with μ_C integrable bosonic coefficients then

$$E_\Lambda(F_\Lambda(\Phi)) = \int d\mu_{C_\Lambda}(\Phi)F_\Lambda(\Phi) = \int d\mu_{C_\Lambda}(\phi)d\mu_{C_\Lambda}(\psi)F_\Lambda(\Phi) \tag{1.21}$$

Thus

$$E_\Lambda(F_\Lambda(\Phi)) = \int \prod_{x\in\Lambda} d\varphi(x)d\bar{\varphi}(x) \prod_{x\in\Lambda} d\psi(x) \prod_{x\in\Lambda} d\bar{\psi}(x)\, e^{-(\Phi, C_\Lambda^{-1}\bar{\Phi})_{L^2(\Lambda)}} F_\Lambda(\Phi) \tag{1.22}$$

Notice that the determinant in the fermionic integration formula (1.14) has cancelled out with the inverse of the same determinant which appears in the bosonic integration measure.

The expectation defined above is normalized. In other words if $1_\Lambda(\Phi)$ is the indicator function of Ω_Λ then

$$E_\Lambda(1_\Lambda(\Phi)) = 1 \tag{1.23}$$

We have the natural order relation $\Omega_\Lambda \subset \Omega_{\Lambda'}$ if $\Lambda \subset \Lambda'$. Moreover if $\Lambda \subset \Lambda'$ and $F_\Lambda \in \Omega_\Lambda$ then $E_{\Lambda'}(F_\Lambda) = E_\Lambda(F_\Lambda)$ as is not difficult to show. We define Ω as the inductive limit of the Ω_Λ as $\Lambda \subset \mathbb{Z}^3$ varies over increasing subsets tending to $\mathbb{Z}^3$ respecting the order relation above. The $\{E_\Lambda, \Omega_\Lambda\}$ constitute a projective family. We denote by E the projective limit: Let $F \in \Omega$ with μ_C integrable bosonic coefficients. We have

$$E(F) = \int d\mu_C(\Phi)\, F(\Phi) = \lim_{\Lambda' \uparrow \mathbb{Z}^3} E_{\Lambda'}(F) \tag{1.24}$$

and this limit exists since $F \in \Omega_\Lambda$ for some finite set Λ and therefore $E(F) = E_\Lambda(F)$ which exists.

Remark The above construction is motivated by analogous considerations in [9].

Lattice Integration In the following and throughout this paper we will represent lattice sums as integrals where for the $(\delta\mathbb{Z})^3$ lattice the integration measure is the counting measure in $(\delta\mathbb{Z})^3$ times a factor δ^3. Thus if f is a function on $(\delta\mathbb{Z})^3$ we define

$$\int_{(\delta\mathbb{Z})^3} dx\, f(x) = \delta^3 \sum_{x\in(\delta\mathbb{Z})^3} f(x) \tag{1.25}$$

We now define a Laplacian acting on functionals in Ω

$$\begin{aligned}\Delta_C &= \int_{\mathbb{Z}^3\times\mathbb{Z}^3} dxdy\, C(x-y)\frac{\partial}{\partial\Phi(x)}\cdot\frac{\partial}{\partial\bar\Phi(y)}\\ &= \int_{\mathbb{Z}^3\times\mathbb{Z}^3} dxdy\, C(x-y)\left[\frac{\partial}{\partial\phi(x)}\frac{\partial}{\partial\bar\phi(y)} + \frac{\partial}{\partial\psi(x)}\frac{\partial}{\partial\bar\psi(y)}\right]\end{aligned} \tag{1.26}$$

These integrals on $\mathbb{Z}^3$ automatically restricts to $\Lambda \times \Lambda$ when applied to functionals of Φ which live in a bounded subset Λ of $\mathbb{Z}^3$. It follows from (1.15) and its bosonic counterpart that if $F_\Lambda(\Phi) \in \Omega^0_\Lambda$ with μ_C integrable bosonic coefficients then

$$E(F_\Lambda(\Phi)) = e^{\Delta_C} F_\Lambda(\Phi)|_{\varphi=\bar\varphi=\psi=\bar\psi=0} \tag{1.27}$$

Note that the action of e^{Δ_C} is well defined. In fact since $F_\Lambda(\Phi)$ is in Ω^0_Λ and Λ is a finite lattice, it can be expressed as a finite sum of Grassmann elements with coefficients in $\mathcal{F}_\Lambda$. e^{Δ_C} factorises into bosonic and Grassmann exponentials. The expansion of the Grassman exponential acting on $F_\Lambda(\Phi)$ evaluated at $\psi = \bar\psi = 0$ thus terminates and we are left with the expectation of the bosonic coefficients which is well defined since they are μ_C integrable by assumption.

We have in particular

$$E(\Phi(x)\bar\Phi(y)) = 0 \tag{1.28}$$

and more generally

$$E\left(\prod_{j=1}^n \Phi(x_j)\bar\Phi(y_j)\right) = 0 \tag{1.29}$$

This can be proved by computation or more simply using supersymmetry (introduced later). The integrand is supersymmetric and Lemma 1.1 below gives the result.

Wick polynomials $P(\Phi)$ are defined by the formula

$$:P(\Phi):_C = e^{-\Delta_C} P(\Phi) \tag{1.30}$$

This implies in particular that

$$:\Phi(x)\bar{\Phi}(y):_C = \Phi(x)\bar{\Phi}(y) \tag{1.31}$$

and

$$:(\Phi\bar{\Phi})^2:_C(x) = (\Phi\bar{\Phi})^2(x) - 2C(0)(\Phi \cdot \bar{\Phi})(x) \tag{1.32}$$

For future reference we note that for $\alpha = 1, 2$

$$:(\Phi\bar{\Phi})\Phi_\alpha:_C(x) = (\Phi\bar{\Phi})\Phi_\alpha(x) - C(0)\Phi_\alpha(x) \tag{1.33}$$

where $\Phi_1 = \varphi,\ \Phi_2 = \psi$.

Remark The considerations from (1.12) to (1.32) remain valid on a lattice $(\delta\mathbb{Z})^3$ if we replace in the above Λ by a bounded subset $\Lambda_\delta \subset (\delta\mathbb{Z})^3$ and the positive definite matrix C by an arbitrary positive definite matrix $C_\delta(x, y)$ with $x, y \in (\delta\mathbb{Z})^3$. The functional Laplacian Δ_C in (1.26) is replaced by Δ_{C_δ} with the integration over $(\delta\mathbb{Z})^3 \times (\delta\mathbb{Z})^3$.

The Model Let L be a triadic integer, $L = 3^p$ with integer $p \geq 2$. Let $\Lambda_N = (-\frac{L^N}{2}, \frac{L^N}{2})^3 \subset \mathbb{R}^3$, with N large be a large open cube in $\mathbb{R}^3$. Distances in $\mathbb{R}^3$ and lattices $(\delta\mathbb{Z})^3$ will be measured in the norm

$$|x - y| = \max_{1 \leq j \leq 3} |x_j - y_j| \tag{1.34}$$

Define $\Lambda_{N,0} = \Lambda_N \cap \mathbb{Z}^3$. This is a (large) cube in $\mathbb{Z}^3$ of edge length L^N. The second index 0 in $\Lambda_{N,0}$ emphasizes that this is a cube in $\mathbb{Z}^3$. The local potential (1.2) will be written in a C-Wick ordered form by using (1.32) and (1.31). This gives

$$V_0(\Lambda_{N,0}, \Phi) = \int_{\Lambda_{N,0}} dx\ g_0 :(\Phi\bar{\Phi})^2:_C(x) + \mu_0 \int_{\Lambda_{N,0}} dx\ :\Phi\bar{\Phi}:_C(x) \tag{1.35}$$

where $\mu_0 = \tilde{\mu}_0 + 2C(0)g_0$.

Define

$$\mathcal{Z}_0(\Lambda_{N,0}, \Phi) = e^{-V(\Lambda_{N,0}, \Phi)} \tag{1.36}$$

We define the measure

$$d\mu_{N,0}(\Phi) = d\mu_C(\Phi)\mathcal{Z}_0(\Lambda_{N,0}, \Phi) \tag{1.37}$$

Note that the measure is normalized

$$\int d\mu_{N,0}(\Phi) = 1 \tag{1.38}$$

This follows from Lemma 1.1 below which exploits supersymmetry introduced later. However heuristically this is evident if we formally expand the exponential, integrate term by term and use (1.29). This measure defines our model.

Supersymmetry

The density of the measure $d\mu_{N,0}(\Phi)$ as well as its RG evolution have the important property of being *supersymmetric*. This will restrict considerably the form of the evolved density.

A *supersymmetry* transformation $\mathcal{Q} : \Omega_\Lambda \to \Omega_\Lambda$ is a derivation on the bosonic fields and an antiderivation on the Grassman fields which acts on the fields as follows:

$$\begin{aligned}
\mathcal{Q}\varphi &= \psi \\
\mathcal{Q}\bar{\varphi} &= -\bar{\psi} \\
\mathcal{Q}\psi &= \varphi \\
\mathcal{Q}\bar{\psi} &= \bar{\varphi}
\end{aligned} \tag{1.39}$$

Let $F_\Lambda(\Phi) = F_\Lambda(\varphi, \bar{\varphi}, \psi, \bar{\psi})$ belong to Ω_Λ with bosonic coefficients differentiable in the bosonic fields $\varphi(x)$, $x \in \Lambda$. Then the action of $\mathcal{Q}$ on F_Λ is given by a super vector field denoted by the same symbol $\mathcal{Q}$

$$\mathcal{Q}F_\Lambda = \int_\Lambda dx \left(\psi(x)\frac{\partial}{\partial\varphi(x)} - \bar{\psi}(x)\frac{\partial}{\partial\bar{\varphi}(x)} + \varphi(x)\frac{\partial}{\partial\psi(x)} + \bar{\varphi}(x)\frac{\partial}{\partial\bar{\psi}(x)}\right) F_\Lambda \tag{1.40}$$

We say that a functional F_Λ is supersymmetric if $\mathcal{Q}F = 0$.

Remark A super vector field is not a vector field because fermionic derivatives are antiderivations.

An (infinitesimal) *gauge transformation* $\mathcal{G} : \Omega_\Lambda \to \Omega_\Lambda$ is a derivation whose action is given by

$$\begin{aligned}
\mathcal{G}\varphi &= i\varphi \\
\mathcal{G}\bar{\varphi} &= -i\bar{\varphi} \\
\mathcal{G}\psi &= i\psi \\
\mathcal{G}\bar{\psi} &= -i\bar{\psi}
\end{aligned} \tag{1.41}$$

This induces on an Ω_Λ function F_Λ the action of a vector field denoted by the same symbol $\mathcal{G}$

$$\mathcal{G}F_\Lambda = i\int_\Lambda dx \left(\varphi(x)\frac{\partial}{\partial\varphi(x)} - \bar{\varphi}(x)\frac{\partial}{\bar{\partial}\varphi(x)} + \psi(x)\frac{\partial}{\partial\psi(x)} + \bar{\psi}(x)\frac{\partial}{\partial\bar{\psi}(x)}\right) F_\Lambda \tag{1.42}$$

We say that a functional F_Λ is gauge invariant if $\mathcal{G}F_\Lambda = 0$.

From (1.39) we see that $\mathcal{Q}^2$ engenders an infinitesimal gauge transformation (1.41). Thus acting on gauge invariant functionals

$$\mathcal{Q}^2 = 0 \tag{1.43}$$

An important property of the super vector field $\mathcal{Q}$ which we will exploit later is that it commutes with the super Laplacian Δ_C defined in (1.26):

$$[\, \mathcal{Q},\ \Delta_C\,] = 0$$

as is easy to verify.

It is easy to verify that any polynomial in $\Phi\bar{\Phi}$ and their (lattice) derivatives is supersymmetric. As a consequence we have $\mathcal{Q}V(\Lambda,\Phi)=0$ where V is given in (1.35) and thus the starting interaction potential is supersymmetric.

Let $\Gamma(x,y)$ be any positive definite symmetric matrix. Let Δ_Γ be a super Laplacian given by (1.26) with C replaced by Γ. Let $F_\Lambda(\Phi)$ be an Ω_Λ functional with μ_C integrable bosonic coefficients. Let $\xi=(\zeta,\eta)$ be another superfield. Define the convolution

$$\mu_\Gamma * F_\Lambda(\Phi)=\int d\mu_\Gamma(\xi)F_\Lambda(\Phi+\xi)=e^{\Delta_\Gamma}F_\Lambda(\Phi) \tag{1.44}$$

Since $\mathcal{Q}$ commutes with Δ_Γ, $\mathcal{Q}$ *also commutes with convolution with the measure* μ_Γ:

$$\mu_\Gamma * \mathcal{Q}F_\Lambda(\Phi)=\mathcal{Q}\mu_\Gamma * F_\Lambda(\Phi) \tag{1.45}$$

Therefore if F_Λ is supersymmetric so is $\mu_\Gamma * F_\Lambda$. This observation prefigures the supersymmetry invariance of the renormalization group map which we will introduce later.

It follows by evaluating (1.45) at $\Phi=0$ that

$$\int d\mu_\Gamma(\Phi)\ \mathcal{Q}F_\Lambda(\Phi)=0 \tag{1.46}$$

since the left hand side is given by $(\mu_\Gamma * \mathcal{Q}F_\Lambda(\Phi))|_{\Phi=0}$ and this vanishes by virtue of (1.45) since the coefficients of the super vector field $\mathcal{Q}$ vanish when the fields vanish.

Lemma 1.1 *Let $F_\Lambda(\Phi)$ be a supersymmtric Ω_Λ functional with differentiable bosonic coefficients which are μ_Γ integrable. Then*

$$\int d\mu_\Gamma(\Phi)\ F_\Lambda(\Phi)=F_\Lambda(0) \tag{1.47}$$

Proof λ be a real parameter. Define

$$f(\lambda)=\int d\mu_\Gamma(\Phi)\ F_\Lambda(\lambda\Phi) \tag{1.48}$$

We will prove

$$\frac{d}{d\lambda}f(\lambda)=0 \tag{1.49}$$

This implies that $f(\lambda)$ is a constant and hence evaluating at $\lambda=0$ gives (1.47).

Taking the λ derivative in (1.48) we get

$$\frac{d}{d\lambda}f(\lambda)=\int d\mu_\Gamma(\Phi)\ (\mathcal{D}F_\Lambda)(\lambda\Phi) \tag{1.50}$$

where

$$\mathcal{D}=\int_\Lambda dx\left(\phi(x)\frac{\partial}{\partial\phi(x)}+\bar{\phi}(x)\frac{\partial}{\partial\bar{\phi}(x)}+\psi(x)\frac{\partial}{\partial\psi(x)}+\bar{\psi}(x)\frac{\partial}{\partial\bar{\psi}(x)}\right) \tag{1.51}$$

Note that the four coefficients of $\mathcal{D}$ can also be written as $(\mathcal{Q}\psi(x),\mathcal{Q}\bar{\psi}(x),\mathcal{Q}\phi(x),-\mathcal{Q}\bar{\phi}(x))$ which we have taken in the same order as above. This suggests that we consider the operator

$$\mathcal{L}=\int_\Lambda dx\left(\psi(x)\frac{\partial}{\partial\phi(x)}+\bar{\psi}(x)\frac{\partial}{\partial\bar{\phi}(x)}+\phi(x)\frac{\partial}{\partial\psi(x)}-\bar{\phi}(x)\frac{\partial}{\partial\bar{\psi}(x)}\right) \tag{1.52}$$

and act with $\mathcal{Q}$ on it. We also consider the action of $\mathcal{L}$ on $\mathcal{Q}$. A straight forward computation gives the nice formula

$$\mathcal{D} = \frac{1}{2}(\mathcal{Q}\mathcal{L} + \mathcal{L}\mathcal{Q}) \tag{1.53}$$

We substitute for $\mathcal{D}$ in (1.50) the right hand side of (1.46). The contribution of the first term vanishes by (1.53). The contribution of the second term vanishes because F_Λ is supersymmetric by hypothesis. This proves (1.50) and we are done. □

Remark The special case of Lemma 1.1 for a hierarchical lattice is Lemma 2.1 of [11]. *This lemma has the important consequence that no field independant relevant parts (defined later) will arise in the renormalization group analysis to follow.*

1.2 Lattice Renormalization Group Transformations

We say that a function $f(x, y)$ has finite range L if $f(x, y) = 0 \ : \ |x - y| \geq L$. Lattice renormalization group transformations will be based on the finite range multiscale expansion of the covariance C established in [8].

Let L be a large triadic integer, $L = 3^p$, $p \geq 2$. Define $\delta_n = L^{-n}$. We have a sequence of compatible lattices $(\delta_n\mathbb{Z})^3 \subset \mathbb{R}^3$, $(\delta_n\mathbb{Z})^3 \subset (\delta_{n+1}\mathbb{Z})^3$, with $n = 0, 1, 2, \ldots$. $B_{\delta_n} = [-\frac{\pi}{\delta_n}, \frac{\pi}{\delta_n}]^3$ denotes the first Brillouin zone of the dual of the δ_n lattice. We have the following theorem which gives the multiscale expansion of the covariance C on $\mathbb{Z}^3$ as a sum of finite range *fluctuation* covariances living on increasingly finer lattices, together with their properties which we will need later:

Theorem 1.1 (Finite range multiscale expansion) *For* $0 < \alpha < 2$, $d_s = \frac{(3-\alpha)}{2}$ *and* $n = 0, 1, 2, \ldots$ *there exist positive definite functions* $\Gamma_n(x)$ *defined for* $x \in (\delta_n\mathbb{Z})^3$ *and a smooth positive definite function* $\Gamma_{c,*}$ *in* $\mathbb{R}^3$ *such that for all* $k \geq 0$, *constants* $c_{k,L}$, $c_{L,m}$ *independent of* n *and* $q = \frac{1}{2}$

(1) $\quad C(x - y) = \sum_{n\geq 0} L^{-2nd_s}\Gamma_n\left(\dfrac{x - y}{L^n}\right)$ and the series converges in $L^\infty(\mathbb{Z}^3)$

(2) $\quad \Gamma_n(x) = 0 \quad$ for $|x| \geq \dfrac{L}{2}$

(3) $\quad \left|\hat{\Gamma}_n(p)\right| \leq c_{k,L}(1 + p^2)^{-2k} \quad$ for $p \in B_{\delta_n}$, $\forall k \geq 0$

(4a) $\quad \hat{\Gamma}_{c,*}(p) = \lim_{n\to\infty} \hat{\Gamma}_n(p) \quad$ exists pointwise in p

(4b) $\quad \left|\hat{\Gamma}_n(p) - \hat{\Gamma}_{c,*}(p)\right| \leq c_{k,L}(1 + p^2)^{-2k}\left(1 + \dfrac{1}{p^2}\right)L^{-qn}, \quad \forall n \geq 3$, $\forall k \geq 0$, $p \in B_{\delta_n} \setminus 0$

(5a) $\quad \|\partial_{\delta_n}^m \Gamma_n\|_{L^\infty((\delta_n\mathbb{Z})^3)} \leq c_{L,m}, \quad \forall m \geq 0$

(5b) $\quad \partial_c^m \Gamma_{c,*} = \lim_{n\to\infty} \partial_{\delta_n}^m \Gamma_n \quad$ exists in $L^\infty((\delta_l\mathbb{Z})^3)$

(5c) $\quad \|\partial_{\delta_n}^m \Gamma_n - \partial_c^m \Gamma_{c,*}\|_{L^\infty((\delta_l\mathbb{Z})^3)} \leq c_{L,m}L^{-qn}, \quad \forall n \geq l \geq 3$, $\forall m \geq 0$

where ∂_c *is a continuum partial derivative,* ∂_{δ_n} *is a forward lattice partial derivative in* $(\delta_n\mathbb{Z})^3$ *and the dependence on the direction vectors have been suppressed. For* $\partial_{\delta_n}^m$ *and* ∂_c^m *a multi-index convention is implicit.*

Remark The theorem is for the most part a combination of results obtained in various theorems in [8]. Before we outline the proof note that in [8], L was a large dyadic integer whereas we have chosen here L to be triadic. The results of [8] remain unaffected provided we define the continuum cube $U_c(R) \subset \mathbb{R}^3$ in Sect. 1 of [8] to be $(-\frac{R}{3}, \frac{R}{3})^3$. This guarantees in particular that if $R = R_m = L^{-(m-1)}$, $0 \le m \le n$, and $U_{\delta_n}(R_m) = U_c(R_m) \cap (\delta_n \mathbb{Z})^3$ then the important property $\partial U_{\delta_n}(R_m) \subset \partial U_c(R_m)$ remains true. This last property is invoked in Sect. 6, p. 439 of [8], in preparation for the convergence proof therein.

Proof The multiscale expansion in part (1) and the finite range property of part (2) were given in Sect. 4 [8]. The factor 6 in the range $6L$ of Γ_n is an artifact. By scaling down R_m in the cube $U_{\delta_n}(R_m)$ by a factor of 3^{-4} and the range of the function g in Sect. 1 to $3^{-6}L$ we get Γ_n to have range $L/2$. Convergence of (1) in $L^\infty(\mathbb{Z}^3)$ follows on using $d_s > 0$, Corollary 5.6 of [8] and the Sobolev embedding inequality for lattice $L^2_k = H_k$ spaces with k in the corollary sufficiently large. Part (3) follows from (5.10) of Theorem 5.5 of [8] by integration on a with the measure $da\, a^{-\alpha/2}$ (see (4.3) of Sect. 4). Corollary 5.6 of [8] and lattice Sobolev embedding gives (5a). Corollary 6.2 of [8] gives parts (4a) and (5b). The convergence rate estimates of parts (4b) and (5c) which were not given in [8] also follow from the results therein. The proof is given elsewhere [12]. □

Remark (4b) is not necessarily the best possible estimate. The left hand side has no singularity at $p = 0$ whereas the right hand side does. However it suffices for our purposes because (5c) above follows from (4b) and it is (5c) which will be put to use later. In fact (4b) implies that for fixed $l \ge 3$ and all $n \ge l$, $k \ge 0$,

$$\|\Gamma_n - \Gamma_{c,*}\|_{L^1_k((\delta_l \mathbb{Z})^3)} \le c_{k,L} L^{-qn}$$

where $L^1_k((\delta_l \mathbb{Z})^3)$ is a lattice Sobolev space. The finite range of Γ_n, $\Gamma_{c,*}$ and lattice Sobolev embedding for $k \ge 3 + m$ implies (5c). The singularity at $p = 0$ in the right hand side of (4b) is integrable in B_{δ_n}. It thus turns out to be harmless.

Define for all $n \ge 0$ the positive definite functions C_n, $C_{c,*}$ on $(\delta_n \mathbb{Z})^3$ and $\mathbb{R}^3$ respectively by the recursion relations

$$C_n(x) = \Gamma_n(x) + L^{-2d_s} C_{n+1}\left(\frac{x}{L}\right) \tag{1.54}$$

$$C_{c,*}(x) = \Gamma_{c,*}(x) + L^{-2d_s} C_{c,*}\left(\frac{x}{L}\right) \tag{1.55}$$

Solving these relations by iteration gives

$$C_n(x) = \sum_{j=0}^{\infty} L^{-2jd_s}\, \Gamma_{n+j}\left(\frac{x}{L^j}\right) \tag{1.56}$$

$$C_{c,*}(x) = \sum_{j=0}^{\infty} L^{-2jd_s}\, \Gamma_{c,*}\left(\frac{x}{L^j}\right) \tag{1.57}$$

Note that $C_0 = C$ as follows from (1) of Theorem 1.1.

Corollary 1.1 *The series* 1.56) *for* C_n *together with that for its multiple lattice derivatives in* $(\delta_n\mathbb{Z})^3$ *converge in* $L^\infty((\delta_n\mathbb{Z})^3)$. *For every integer* $m\geq 0$ *we have a constant* $c_{L,m}$ *such that*

$$\|\partial^m_{\delta_n} C_n\|_{L^\infty((\delta_n\mathbb{Z})^3)} \leq c_{L,m} \tag{1.58}$$

The series (1.56) *defining* $C_{c,*}$ *and its multiple continuum derivatives of arbitrary order converge in* $L^\infty(\mathbb{R}^3)$ *so that* $C_{c,*}$ *is a smooth continuum function. For all* $m\geq 0$ *and* ∂_c *the continuum partial derivative*

$$\sup_{x\in\mathbb{R}^3} |\partial^m_c C_{c,*}(x)| \leq c_{L,m} \tag{1.59}$$

Moreover for $n\geq l\geq 3$ *with* l *fixed and* $\forall m\geq 0$, *there exists a constant* $c_{L,m}$ *such that*

$$\|\partial^m_{\delta_l} C_n - \partial^m_c C_{c,*}\|_{L^\infty((\delta_l\mathbb{Z})^3)} \leq c_{L,m} L^{-qn} \tag{1.60}$$

Proof The first part together with the bound (1.58) follow from (5a) of Theorem 1.1. In fact from (1.56) we have

$$\partial^m_{\delta_n} C_n = \sum_{j=0}^{\infty} L^{-2jd_s}\ L^{-mj} (\partial^m_{\delta_{n+j}} \Gamma_{n+j})\left(\frac{x}{L^j}\right)$$

where we have used repeatedly (m-times) the identity $\partial_{\delta_n}\Gamma_{n+j}(\frac{x}{L^j}) = L^{-j}(\partial_{\delta_{n+j}}\Gamma_{n+j})(\frac{x}{L^j})$ as is easy to show. Therefore

$$\begin{aligned}
\|\partial^m_{\delta_n} C_n\|_{L^\infty((\delta_n\mathbb{Z})^3)} &\leq \sum_{j=0}^{\infty} L^{-2jd_s} L^{-mj} \sup_{x\in(\delta_n\mathbb{Z})^3} \left|(\partial^m_{\delta_{n+j}}\Gamma_{n+j})\left(\frac{x}{L^j}\right)\right| \\
&\leq \sum_{j=0}^{\infty} L^{-2jd_s} L^{-mj} \sup_{y\in(\delta_{n+j}\mathbb{Z})^3} \left|(\partial^m_{\delta_{n+j}}\Gamma_{n+j})(y)\right|
\end{aligned}$$

Now use the bound in (5a) together with $d_s\geq \frac{1}{2}$ to get (1.58). To prove the next statement observe that the first part of Theorem 6.1 of [8] together with Sobolev embedding implies that $\|\partial^m_c\Gamma_{c,*}\|_{L^\infty(\mathbb{R}^3)} \leq c_{L,m}$. Using this (1.59) follows from (1.57). Finally to prove the estimate (1.60) observe that

$$\begin{aligned}
\|\partial^m_{\delta_l} C_n - \partial^m_c C_{c,*}\|_{L^\infty((\delta_l\mathbb{Z})^3)} &\leq \sum_{j=0}^{\infty} L^{-2jd_s} L^{-mj} \sup_{x\in(\delta_l\mathbb{Z})^3} \left|(\partial^m_{\delta_{n+j}}\Gamma_{n+j})\left(\frac{x}{L^j}\right) - (\partial^m_c C_{c,*})\left(\frac{x}{L^j}\right)\right| \\
&\leq \sum_{j=0}^{\infty} L^{-2jd_s} L^{-mj} \sup_{y\in(\delta_{l+j}\mathbb{Z})^3} \left|(\partial^m_{\delta_{n+j}}\Gamma_{n+j})(y) - \partial^m_c C_{c,*}(y)\right| \\
&\leq L^{-nq} c_{L,m} \sum_{j=0}^{\infty} L^{-2jd_s} L^{-mj} L^{-jq}
\end{aligned}$$

where in the last line we have used part (5c) of Theorem 1.1. Equation (1.60) now follows with a new constant $c_{L,m}$. This also establishes that $\partial^m_{\delta_l} C_n \to \partial^m_c C_{c,*}$ in $L^\infty((\delta_l\mathbb{Z})^3)$. □

We consider the finite sequence of compatible lattices $\{(\delta_n\mathbb{Z})^3\}$ for $0\leq n\leq N$. The considerations in Sect. 1.1 for fields in $\mathbb{Z}^3$ remain valid for every lattice $(\delta_n\mathbb{Z})^3$ provided for the expectations we replace the covariance C by C_n. Let the fields φ, ψ, $\bar{\psi}$ be defined in $(\delta_N\mathbb{Z})^3$. These fields restrict to the coarser lattices $(\delta_n\mathbb{Z})^3$ for every n with $0 \le n \le N$.

We introduce a parameter ε with $0 < \varepsilon \le 1$ and define

$$\alpha = \frac{3+\varepsilon}{2} \tag{1.61}$$

Let $x \in (\delta_n\mathbb{Z})^3$. For every $n \le N-1$ we define the scale transformation S_L by

$$S_L\Phi(x) = \Phi_{L^{-1}}(x) = L^{-d_s}\Phi\left(\frac{x}{L}\right) \tag{1.62}$$

where

$$d_s = \frac{(3-\alpha)}{2} = \frac{3-\varepsilon}{4} \tag{1.63}$$

is the *dimension* of the field Φ. The fields $\varphi, \bar{\varphi}, \psi, \bar{\psi}$ are thus assigned the same dimension d_s and the same transformation law 1.62. Note that the scale transformed fields now live in $(\delta_n\mathbb{Z})^3$.

Let $\Lambda \subset \mathbb{R}^3$ and $\Lambda_{\delta_n} = \Lambda \cap (\delta_n\mathbb{Z})^3$. We define the scale transformation on functionals of fields by

$$(S_L F)(L^{-1}\Lambda_{\delta_{n+1}}, \Phi) = F(\Lambda_{\delta_n}, S_L\Phi) \tag{1.64}$$

The C_n and Γ_n are positive definite and therefore qualify as covariances of Gaussian measures. For $x, y \in (\delta_n\mathbb{Z})^3$ we define the scale transformation of the covariance C_{n+1} by

$$S_L C_{n+1}(x-y) = L^{-2d_s}C_{n+1}\left(\frac{x-y}{L}\right) \tag{1.65}$$

which permits us to write 1.54 as

$$C_n(x-y) = \Gamma_n(x-y) + S_L C_{n+1}(x-y) \tag{1.66}$$

Let $\Lambda_{\delta_n} \subset (\delta_n\mathbb{Z})^3$ be a bounded subset. Then (1.66) implies upon using (1.27) (with C replaced by C_n) that

$$\int d\mu_{C_n}(\Phi)F(\Lambda_{\delta_n}, \Phi) = \int d\mu_{S_L C_{n+1}}(\Phi)\int d\mu_{\Gamma_n}(\xi)F(\Lambda_{\delta_n}, \xi + \Phi) \tag{1.67}$$

Let $L = 3^p$ with integer $p \ge 2$ and let $\Lambda_m = (-\frac{L^m}{2}, \frac{L^m}{2})^3 \subset \mathbb{R}^3$ be an open cube in $\mathbb{R}^3$ centered at the origin. We denote by

$$\Lambda_{m,n} = \Lambda_m \cap (\delta_n\mathbb{Z})^3 \tag{1.68}$$

the induced cube of side length L^m in $(\delta_n\mathbb{Z})^3$ centered at the origin. Let $F_0(\Lambda_{N,0}, \Phi)$ be a functional of Φ and $(\bar{\Phi})$ belonging to $\Omega^0(\Lambda_{N,0})$. By virtue of (1.67) we have for $n = 0$

$$\int d\mu_{C_0}(\Phi)F_0(\Lambda_{N,0}, \Phi) = \int d\mu_{C_1}(\Phi)F_1(\Lambda_{N-1,1}, \Phi) \tag{1.69}$$

where

$$F_1(\Lambda_{N-1,1}, \Phi) := (S_L\mu_{\Gamma_0} * F_0)(\Lambda_{N-1,1}, \Phi) = \int d\mu_{\Gamma_0}(\xi)F_0(\Lambda_{N,0}, \xi + S_L\Phi) \tag{1.70}$$

The final scale transformation takes us to a finer lattice as well as scaling down the size of the cube.

The iteration of (1.70) using (1.69) gives after n steps

$$\int d\mu_{C_0} F_0(\Lambda_{N,0}\,,\Phi) = \int d\mu_{C_n} F_n(\Lambda_{N-n,n}\,,\Phi) \tag{1.71}$$

where

$$F_n(\Lambda_{N-n,n}\,,\Phi) := \mu_{\Gamma_{n-1}} * F_{n-1}(\Lambda_{N-n+1,n-1},\ S_L\Phi) \tag{1.72}$$

Equation (1.72) defines for $N > 0$ fixed and $1 \leq n \leq N-1$ a sequence of maps

$$T_{N-n,n}\ :\ \Omega^0(\Lambda_{N-n+1,n-1}) \to \Omega^0(\Lambda_{N-n,n}) \tag{1.73}$$

any member of which we call a *renormalization group* (*RG*) *transformation*. The map is clearly not autonomous. The first index refers to the cube whose size has gotten reduced because of the rescaling. The second index refers to the lattice spacing which has gotten finer because of the rescaling. In the following we will apply the RG transformation iteratively to the (interaction) density $\mathcal{Z}_0(\Lambda_{N,0}\,,\Phi)$ of the measure $d\mu_{N,0}(\Phi)$ defined in (1.37) generating thereby the sequence $\mathcal{Z}_n(\Lambda_{N-n,n},\Phi)$ for $0 \leq n \leq N-1$. After $N-1$ steps we arrive at $\mathcal{Z}_{N-1}(\Lambda_{1,N-1},\Phi)$ where $\Lambda_{1,N-1}$ is the cube of edge length L in $(\delta_{N-1}\mathbb{Z})^3$ centered at the origin. The fundamental goal in this paper is to control this sequence of transformations when N is indefinitely large in the infinite volume limit (as explained at the end of Sect. 3).

1.3 Polymer Gas Representation

In order to analyze the RG evolution we will write the densities $\mathcal{Z}_n$ in a *polymer gas* representation whose form is preserved under RG transformations.

Polymers We pave $\mathbb{R}^3$ with a disjoint union of open cubes $\Delta \subset \mathbb{R}^3$ of edge length 1 called unit cubes or 1-cubes defined by

$$\Delta = \left(-\frac{1}{2}+m_1, \frac{1}{2}+m_1\right) \times \left(-\frac{1}{2}+m_2, \frac{1}{2}+m_2\right) \times \left(-\frac{1}{2}+m_3, \frac{1}{2}+m_3\right) \tag{1.74}$$

where $(m_1, m_2, m_3) \in \mathbb{Z}^3$. *We say two unit cubes from the paving are connected if their closures share at least a vertex in common.* If they are not connected (i.e. their closures are disjoint) we say that they are *strictly disjoint*. A continuum (connected) 1-polymer X is a (connected) union of a finite subset of unit cubes chosen from the paving and is thus open. Henceforth, unless otherwise mentioned, a polymer is connected by default.

We will measure distances in $\mathbb{R}^3$ and in all embedded lattices in the norm

$$|x-y| = \max_{1\leq j\leq 3} |x_j - y_j| \tag{1.75}$$

If Δ_1 and Δ_2 are two unit cubes from the paving then the distance between them is

$$d(\Delta_1,\Delta_2) = \inf_{x\in\Delta_1,\ y\in\Delta_2} |x-y| \tag{1.76}$$

If Δ_1 and Δ_2 are strictly disjoint than $d(\Delta_1,\Delta_2) \geq 1$.

Let $\delta_n = L^{-n}$ where $L = 3^p$ is a triadic integer. Let δ be any member of the sequence $\{\delta_n\}_{n\geq 0}$. Define the *unit block or* 1*-block* in $(\delta\mathbb{Z})^3$ by

$$\Delta_\delta = \Delta \cap (\delta\mathbb{Z})^3 \tag{1.77}$$

and the *lattice* 1*-polymer* X_δ by

$$X_\delta = X \cap (\delta\mathbb{Z})^3 \tag{1.78}$$

where X is a continuum 1-polymer. Note that as point sets $X_{\delta_n} \subset X_{\delta_{n+1}}$.

We denote by $|X_\delta|$ the volume of X_δ measured in accordance with (1.25). The 1-blocks are lattice restrictions of the open continuum unit cubes defined above. Therefore, as is easy to verify, $|\Delta_{\delta_n}| = 1$ and

$$|X_{\delta_n}| = \sharp\{\Delta_{\delta_n} \,:\, \Delta_{\delta_n} \subset X_\delta\} \tag{1.79}$$

the total number of 1-blocks in X_{δ_n}. This is equal to $|X|$ the total number of 1-cubes in X by our construction. As a consequence we have $|X_{\delta_n}| = |X_{\delta_{n+1}}|$.

We say two 1*-blocks in* X_δ *are connected if the continuum* 1*-cubes of which they are the lattice restrictions are connected* (*see above*). If the 1-blocks are not connected we say that they are *strictly disjoint*. The distance between two strictly disjoint 1-blocks is ≥ 1. The lattice (connected) polymer X_δ is a (connected) union of a finite subset of disjoint 1-blocks Δ_δ. Let X_δ and Y_δ be each a connected polymer. *We say that* X_δ, Y_δ *are strictly disjoint if they are mutually disconnected i.e. if every* 1*-block from* X_δ *is strictly disjoint from every* 1*-block from* Y_δ. Then the distance $d(X_\delta, Y_\delta) \geq 1$.

Given an integer $n \geq 1$ we define the *n-collar* of X_δ, denoted $\partial_n X_\delta$ by

$$\partial_n X_\delta = \{y \notin X_\delta \,: |x-y| \leq n\delta, \text{ some } x \in X_\delta\} \tag{1.80}$$

where $|\cdot|$ is the distance function inherited from $\mathbb{R}^3$. We define

$$\tilde{X}_\delta^{(n)} = X_\delta \cup \partial_n X_\delta \tag{1.81}$$

Let $f : (\delta\mathbb{Z})^3 \to \mathbb{C}$. We define the forward lattice partial derivative $\partial_{\delta,\mu}$ and the backward lattice derivative $\partial_{\delta,-\mu}$ by

$$\partial_{\delta,\mu} f(x) = \delta^{-1}(f(x+\delta e_\mu) - f(x)) \tag{1.82}$$

$$\partial_{\delta,-\mu} f(x) = \partial^*_{\delta,\mu} f(x) = \delta^{-1}(f(x-\delta e_\mu) - f(x)) \tag{1.83}$$

where e_1, e_2, e_3 is the standard basis of unit vectors which provides the orientation of $\mathbb{R}^3$ and thus of all the embedded lattices we will encounter. $\partial^*_{\delta,\mu}$ is the $L^2((\delta\mathbb{Z})^3)$ adjoint of $\partial_{\delta,\mu}$.

Polymer Activity A *polymer activity* $K(X_\delta, \Phi) = \tilde{K}(X_\delta, \varphi, \psi)$, where it is henceforth understood that it also depends on $\bar{\varphi}, \bar{\psi}$, is a map $X_\delta,\ \Phi \to \Omega^0_{\tilde{X}_\delta^{(2)}}$ where the fields Φ depend only on the points of $\tilde{X}_\delta^{(2)}$.

The polymer activities of this paper are of degree 0*, gauge invariant and supersymmetric, and invariant under translations, reflections and rotations which leave the lattice invariant. In addition they satisfy the condition* $K(X_\delta, \Phi) = K(X_\delta, -\Phi)$ *together with the support condition*: $K(X_\delta, \Phi) = 0$ *if* X *is not connected. Furthermore* $K(X_\delta, 0) = 0$.

We write the generic density $\mathcal{Z}(\Lambda_\delta)(\Phi)$ in the form

$$\mathcal{Z}(\Lambda_\delta) = \sum_{N=0}^{\infty} \frac{1}{N!} e^{-V(X_\delta^{(c)})} \sum_{X_{\delta,1},\dots,X_{\delta,N}} \prod_{j=1}^{N} K(X_{\delta,j}) \tag{1.84}$$

where the connected polymers $X_{\delta,j} \subset \Lambda_\delta$ are strictly disjoint, $X_\delta = \bigcup_1^N X_{\delta,j}$, $X_\delta^{(c)} = \Lambda_\delta \backslash X_\delta$ and $V(Y_\delta) = V(Y_\delta, \Phi, C, g, \mu)$ is given by (1.35) with parameters g, μ and integration over Y_δ with measure dx defined as the counting measure in $(\delta\mathbb{Z})^3$ times δ^3. The Wick ordering covariance $C = C_n$ (see (1.56)) if $\delta = \delta_n$. We have suppressed the field dependence in (1.84). Initially the activities K vanish but they do arise under RG transformations. The representation (1.84) remains stable under RG transformations as we will see in Sect. 3.

Polymer activities $K(X_\delta, \Phi) = K(X_\delta, \varphi, \psi) \in \Omega^0_{\tilde{X}_\delta^{(2)}}$ can be represented uniquely as a (finite) series in the fermionic fields $\psi, \bar{\psi}$ with coefficients which are functionals of the bosonic fields φ:

$$K(X_\delta, \Phi) = K(X_\delta, \varphi, \psi) = \sum_{p \ge 0} \frac{1}{(p!)^2} \int_{X_\delta^p \times X_\delta^p} d\mathbf{x} d\mathbf{y}\, (D_F^{2p} K)(X_\delta, \varphi, \mathbf{x}, \mathbf{y}) \prod_{j=1}^{p} \psi(x_j)\bar{\psi}(y_j) \tag{1.85}$$

where $\mathbf{x} = (x_1, \dots, x_p)$, $\mathbf{y} = (y_1, \dots, y_p)$ and $d\mathbf{x} = \prod_{i=1}^{2p} dx_i$ where dx_i is the counting measure multiplied by δ^3 on $(\delta\mathbb{Z})^3$. $\mathbf{y}$ and $d\mathbf{y}$ are similarly defined. The coefficient $(D_F^{2p} K)(X_\delta, \varphi, \mathbf{x}, \mathbf{y})$ is defined by

$$(D_F^{2p} K)(X_\delta, \varphi, \mathbf{x}, \mathbf{y}) = \prod_{j=0}^{p-1} \frac{\partial}{\partial \bar{\psi}(y_{p-j})} \frac{\partial}{\partial \psi(x_{p-j})} K(X_\delta, \varphi, \psi)\Big|_{\psi=\bar{\psi}=0} \tag{1.86}$$

This defines a lattice analogue of a distributional kernel which is henceforth restricted so as to contain at most (*lattice*) *delta functions and their first and second* (*lattice*) *derivatives.* It is clearly antisymmetric in $(x_1, \dots, x_p)$ and in $(y_1, \dots, y_p)$. It is gauge invariant as is the Grassmann monomial of degree 0.

The polymer activities in question also satisfy

$$K(X_\delta, 0) = 0 \tag{1.87}$$

Remarks We will see that the representations (1.84), (1.85) are preserved by renormalization group transformations. *The RG transformations are gauge invariant, preserve supersymmetry by virtue of* (1.45), *as well as the vanishing condition* (1.87) *by virtue of Lemma 1.1. The RG transformations preserve invariance of the polymer activities under translations, reflections and rotations which leave the lattice invariant.*

2 Regulators, Derivatives and Norms

In this section we will introduce Banach spaces of polymer activities. These are lattice analogues of the continuum constructions in [1, 6, 13] albeit with changes because of the presence of Grassman variable. The Banach space norms that we will presently introduce measure differentiability properties of the activities with respect to fields φ, ψ, as well as the behaviour with respect to large fields $\partial\varphi$ and large sets. The behaviour for large φ itself will be controlled with the help of lattice Sobolev inequalities and the local potential.

2.1 Regulators

Let $\partial_{\delta,\mu}$ and $\partial_{\delta,-\mu}$ be respectively the forward and backward lattice derivatives in $(\delta\mathbb{Z})^3$ along the unit vector e_μ defined in (1.82) and (1.83). Here as before δ is any member of the sequence $\{\delta_n\}$ where $\delta_n = L^{-n}$ and $L = 3^p$ with integer $p \geq 2$. Define

$$\partial^j_{\mu_1,\mu_2,\ldots,\mu_j} = \partial_{\delta,\mu_1}\partial_{\delta,\mu_2}\cdots\partial_{\delta,\mu_j}$$

Let X be a connected polymer in $\mathbb{R}^3$ and $X_\delta = X \cap (\delta\mathbb{Z})^3$. Let $\tilde{X}^{(n)}_\delta = X_\delta \cup \partial_n X_\delta$ as defined earlier (1.80) and (1.81)). Let $\varphi : \tilde{X}^{(5)}_\delta \to \mathbb{C}$. We define a norm $\|\cdot\|_{X_\delta,1,5}$:

$$\|\varphi\|^2_{X_\delta,1,5} = \sum_{j=1}^{5} \frac{1}{2^j} \sum_{\mu_j\in S,\forall j} \int_{X_\delta} dx\, |\partial^j_{\mu_1,\mu_2,\ldots,\mu_j}\varphi(x)|^2 \tag{2.1}$$

where $S = \{1,-1,2,-2,3,-3\}$. This is a lattice Sobolev norm of the type introduced in Sect. 5, p. 421 of [8] but now without the L^2 piece.

We define now the *large field regulator*

$$G_\kappa \,:\, X_\delta \times \mathcal{F}_{\tilde{X}^{(5)}_\delta} \to \mathbb{R} \tag{2.2}$$

where $\mathcal{F}_{\tilde{X}^{(5)}_\delta}$ is the algebra of $\mathbf{C}$ valued functions on $\tilde{X}^{(5)}_\delta$ by

$$G_\kappa(X_\delta, \varphi) = e^{\kappa\|\varphi\|^2_{X_\delta,1,5}} \tag{2.3}$$

G_κ satisfies the multiplicative property: If X_δ, Y_δ are disjoint sets then

$$G_\kappa(X_\delta \cup Y_\delta, \varphi) = G_\kappa(X_\delta,\varphi)G_\kappa(Y_\delta,\varphi) \tag{2.4}$$

G^{-1}_κ will be a weight function in polymer activity norms. The norm $\|\cdot\|_{X_\delta,1,5}$ can be used in lattice Sobolev inequalities, in conjunction with the stability provided by the local potential, to control φ and its first two lattice derivatives pointwise. The parameter $\kappa = \kappa(L) > 0$ is chosen so that for all $L \geq 2$ the large field regulator satisfies the stability property given in the following Lemma:

Lemma 2.1 (Stability property) *There exists a constant $\kappa_0 = \kappa_0(L) > 0$ independent of n such that for all κ with $0 < \kappa \leq \kappa_0$*

$$\int d\mu_{\Gamma_n}(\zeta)\, G_\kappa(X_{\delta_n}, \zeta + \varphi) \leq 2^{|X_\delta|} G_{2\kappa}(X_{\delta_n}, \varphi) \tag{2.5}$$

where $|X_{\delta_n}|$ is the number of unit blocks in X_{δ_n}.

Proof Equation (2.5) is proved in exactly the same way as in the proof of the stability property of the continuum large field regulator in Lemma 3 of [6]. The proof uses a flow equation for the measure convolution with interpolated covariance which remains true for the lattice. Another ingredient is Young's convolution inequality for functions which is also true on the lattice. In the cited proof we replace the covariance C by Γ_n and continuum derivatives by lattice derivatives. From the proof of Lemma 3 of [6] we see that two conditions have to be satisfied by κ_0, namely: (1) $\kappa_0 \max_{2\leq m\leq 10} \|\partial^m_{\delta_n}\Gamma_n\|_{L^\infty((\delta_n\mathbf{Z})^3)}$ is sufficiently small and (2)

$\kappa_0\|\Gamma_n\|_{L^1((\delta_n\mathbf{Z})^3)}$ is sufficiently small. Parts (5a) of Theorem 1.1 shows that (1) and (2) above can be assured by a κ_0 independent of n. From (5a) we have

$$\kappa_0 \max_{2\le m\le 10}\|\partial_{\delta_n}^m\Gamma_n\|_{L^\infty((\delta_n\mathbf{Z})^3)}\le\kappa_0 c_L$$

and from (5a) and the finite range property

$$\kappa_0\|\Gamma_n\|_{L^1((\delta_n\mathbf{Z})^3)}\le\kappa_0 L^3 c'_L$$

It is sufficient to choose κ_0 so that the right hand side of both inequalities are sufficiently small. This is achieved independent of n. □

Now hold $L=3^p$ sufficiently large by taking p large. Recall that $\alpha=\frac{3+\varepsilon}{2}$ where $0<\varepsilon<1$ so that $\alpha<2$. Then we get after rescaling

$$\int d\mu_\Gamma(\zeta)\,G_\kappa(X_{\delta_n},\zeta+S_L\varphi)\le 2^{|X_{\delta_n}|}G_\kappa(L^{-1}X_{\delta_{n+1}},\varphi)\tag{2.6}$$

because from the scaling property of the fields φ, see (1.62), (1.63) we have

$$\|S_L\varphi\|^2_{X_{\delta_n},1,5}\le L^{-(2-\alpha)}\|\varphi\|^2_{L^{-1}X_{\delta_{n+1}},1,5}\tag{2.7}$$

Next we introduce a *large set regulator*. Let X_δ be a connected 1-polymer in $(\delta\mathbb{Z})^3$. This is a connected union of 1-blocks defined earlier. We define

$$\mathcal{A}_p(X_\delta)=2^{p|X_\delta|}L^{(D+2)|X_\delta|}\tag{2.8}$$

where for us the dimension of space $D=3$, and p is an integer.

Small sets: We call a connected polymer X_δ *small* if $|X_\delta|\le 2^D$. A connected polymer which is not small is called *large*.

L-polymers and L-closure: Pave $\mathbb{R}^3$ by a disjoint union of open cubes $L\Delta$ of edge length L, called L-cubes:

$$L\Delta=\left(-\frac{L}{2}+m_1L,\frac{L}{2}+m_1L\right)\times\left(-\frac{L}{2}+m_2L,\frac{L}{2}+m_2L\right)\times\left(-\frac{L}{2}+m_3L,\frac{L}{2}+m_3L\right)\tag{2.9}$$

where $(m_1,m_2,m_3)\in\mathbb{Z}^3$. Each L-cube is a union of 1-cubes. Let δ be any member of the sequence $\{\delta_n\}_{n\ge0}$ where $\delta_n=L^{-n}$, $L=3^p$ and $p\ge2$. Take the restriction of these L-cubes to $(\delta\mathbb{Z})^3$ and call the latter cubes L-blocks. Each L-block is a union of 1-blocks. The paving of $\mathbb{R}^3$ by L-cubes induces a paving of $(\delta\mathbb{Z})^3$ by L-blocks. An L-polymer is a union of L- blocks. *We define the L-closure of the 1-polymer X_δ, denoted $\bar{X_\delta}^{(L)}$, as the L-polymer given by the smallest union of L-blocks containing X_δ.* The notions of connectedness and strict disjointness carry over from the case of 1 blocks and 1-polymers. Thus we say two L-blocks from the L-paving are connected if the closures of the corresponding continuum L-cubes are connected (i.e. share at least a vertex in common). If they are not connected we say that they are strictly disjoint. Strictly disjoint L-blocks are separated by a distance $\ge L$. A connected L-polymer is a connected union of L-blocks. If two connected L-polymers are not connected to each other we say they are strictly disjoint. Strictly disjoint L-polymers are separated by a distance $\ge L$.

Lemma 2.2 *Fix any integer $p \geq 0$ and let L be sufficiently large depending on p. Then for any connected 1-polymer X_δ*

$$\mathcal{A}(L^{-1}\bar{X}_\delta{}^{(L)}) \leq c_p \mathcal{A}_{-p}(X_\delta) \tag{2.10}$$

For X_δ a large connected 1-polymer,

$$\mathcal{A}(L^{-1}\bar{X}_\delta{}^{(L)}) \leq c_p L^{-D-1} \mathcal{A}_{-p}(X_\delta) \tag{2.11}$$

Here $c_p = O(1)$ is a constant independent of L and δ.

Remark This is the lattice version of Lemma 1 of [6]. It is purely geometrical and proved in the same way.

2.2 Field Derivatives and Norms

The polymer activities in question are degree 0 gauge invariant supersymmetric functionals of the complex bosonic fields $\varphi, \bar{\varphi}$ and the fermionic fields $\psi, \bar{\psi}$. Lattice field derivatives are partial derivatives with respect to the fields at different points of the lattice. The fermionic derivative is an antiderivation. However in order to measure the size of the lattice field derivatives it turns out to be useful to generalize the notion of field derivatives as directional derivatives (directional in field space). For the bosonic coefficient this is the lattice transcription of that given in [6]. For the fermionic part there is no clear sense of direction and the definition we give below suggested to us by David Brydges is both natural and useful.

Let $X_\delta \subset (\delta\mathbb{Z})^3$ be a connected polymer. Let f_j for $j = 1, \dots, m$ be $\mathbf{C}$ valued functions on $\tilde{X}_\delta^{(2)}$. Let $g_{2p}(\mathbf{x}, \mathbf{y}) =: g_{2p}(x_1, \dots, x_p, y_1, \dots, y_p)$ be a $\mathbf{C}$ valued function on $(\tilde{X}_\delta^{(2)})^p \times (\tilde{X}_\delta^{(2)})^p$, antisymmetric in the x_j and in the y_j. A polymer activity $K(X_\delta, \Phi)$ has the representation (1.85) with the coefficients defined in (1.86). We consider it as a function of $\varphi, \bar{\varphi}, \psi, \bar{\psi}$ denoted as $K(X_\delta, \varphi, \psi)$ where we have suppressed the dependence on $\bar{\varphi}, \bar{\psi}$. We define using the notations of (1.85), (1.86) for the coefficients,

$$\begin{aligned} &D^{2p,m} K(X_\delta, \varphi, 0; f^{\times m}, g_{2p}) \\ &\quad =: \int_{X_\delta^p \times X_\delta^p} d\mathbf{x} d\mathbf{y} D_B^m D_F^{2p}(X_\delta, \varphi, \mathbf{x}, \mathbf{y}; f^{\times m}) g_{2p}(x_1, \dots, x_p, y_1, \dots, y_p) \end{aligned} \tag{2.12}$$

where $f^{\times m} = (f_1, \dots, f_m)$ and

$$\begin{aligned} &D_B^m D_F^{2p} K_{2p}(X_\delta, \varphi, \mathbf{x}, \mathbf{y}; f^{\times m}) \\ &\quad = \partial_{s_1} \cdots \partial_{s_m} D_F^{2p} K(X_\delta, \varphi + s_1 f_1, \dots, \varphi + s_m f_m, \mathbf{x}, \mathbf{y})|_{s_1 = \dots = s_m = 0} \end{aligned} \tag{2.13}$$

and the s_j are real parameters.

Let $\partial_{\delta,\mu}$, $\partial_{\delta,-\mu}$ be the forward and backward lattice derivative in the direction e_μ. Let the index set S be defined as after (2.1). We endow the linear space of $\mathbf{C}$ valued functions f as above with the norm

$$\|f\|_{C^2(X_\delta)} = \sup_{\mu,\nu \in S} (\|f\|_{L^\infty(X_\delta)}\|,\ \|\partial_{\delta,\mu} f\|_{L^\infty(X_\delta)},\ \|\partial_{\delta,\mu}\partial_{\delta,\nu} f\|_{L^\infty(X_\delta)}) \tag{2.14}$$

and call the resulting normed space $C^2(X_\delta)$.

Let ∂_{δ,μ_j}, $\mu_j \in S$, acting on $g_{2p}(\mathbf{x},\mathbf{y})$ denote the forward or backward lattice derivative with respect to x_j or y_j in the direction e_{μ_j}. We endow the linear space of $\mathbf{C}$ valued functions $g_{2p}(\mathbf{x},\mathbf{y})$ on $(\tilde{X}_\delta^{(2)})^p \times (\tilde{X}_\delta^{(2)})^p$, antisymmetric in the x_j, and in the y_j, with the norm

$$\|g_{2p}\|_{C^2(X_\delta^{2p})} = \sup_{\substack{\mu_j,\mu_k\in S\\ 1\le j,k\le 2p}} (\|g_{2p}\|_{L^\infty(X_\delta^{2p})}, \ \|\partial_{\delta,\mu_j} g_{2p}\|_{L^\infty(X_\delta^{2p})}, \ \|\partial_{\delta,\mu_j}\partial_{\delta,\mu_k} g_{2p}\|_{L^\infty(X_\delta^{2p})}) \quad (2.15)$$

and call the resulting normed space $C_a^2(X_\delta^{2p})$. The above norms always exist for lattice functions since X_δ is a finite set.

Equation (2.12) then defines a $\mathbf{C}$ valued multilinear functional on $C^2(X_\delta)^m \times C_a^2(X_\delta^{2p})$ whose norm is defined to be

$$\|D^{2p,m}K(X_\delta,\varphi,0)\| = \sup_{\substack{\|f_j\|_{C^2(X_\delta)}\le 1\\ \|g_{2p}\|_{C^2(X_\delta^{2p})}\le 1\\ \forall 1\le j\le m}} \left|D^{2p,m}K(X_\delta,\varphi,0;f^{\times m},g_{2p})\right| \quad (2.16)$$

The space of $\mathbf{C}$ valued multilinear functionals defined in (2.12) which are bounded in the norm (2.16) is complete and thus a Banach space.

Remark It is well known that the space of bounded $\mathbf{C}$ valued multilinear functionals on a normed space is complete (even if the normed space is not). The completeness follows on using the completeness of the number field $\mathbf{C}$ by a standard argument.

Let $\mathbf{h} = (h_F, h_B)$ where $h_F, h_B > 0$ are strictly positive real numbers. We define the following set of norms. The $\mathbf{h}$ norm is defined by

$$\|K(X_\delta,\varphi,0)\|_{\mathbf{h}} = \sum_{p=0}^{\infty}\sum_{m=0}^{m_0} \frac{h_F^{2p}}{(p!)^2}\frac{h_B^m}{m!}\|D^{2p,m}K(X_\delta,\varphi,0)\| \quad (2.17)$$

In addition we define a *kernel* norm with $\mathbf{h}_* = (h_F, h_{B*})$

$$|K(X_\delta)|_{\mathbf{h}_*} = \sum_{p=0}^{\infty}\sum_{m=0}^{m_0} \frac{h_F^{2p}}{(p!)^2}\frac{h_{B*}^m}{m!}\|D^{2p,m}K(X_\delta,0,0)\| \quad (2.18)$$

$\mathbf{h}, \mathbf{h}_*$ will be chosen later in Sect. 5. We now define the $\mathbf{h}$, G_κ norm by

$$\|K(X_\delta)\|_{\mathbf{h},G_\kappa} = \sup_{\varphi\in\mathcal{F}_{\tilde{X}_\delta^{(5)}}} \|K(X_\delta,\varphi,0)\|_{\mathbf{h}} G_\kappa^{-1}(X_\delta,\varphi) \quad (2.19)$$

Let $\mathcal{A}(X_\delta)$ be the large set regulator defined earlier. We then have our final set of norms

$$\|K\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta} = \sup_{\Delta_\delta}\sum_{X_\delta\supset\Delta_\delta} \|(K(X_\delta)\|_{\mathbf{h},G_\kappa}\mathcal{A}(X_\delta) \quad (2.20)$$

where $\Delta_\delta = \Delta\cap(\delta\mathbb{Z})^3$ and Δ is a unit cube in $\mathbb{R}^3$ as defined earlier, and

$$|K|_{\mathbf{h}_*,\mathcal{A},\delta} = \sup_{\Delta_\delta}\sum_{X_\delta\supset\Delta_\delta} |K(X_\delta)|_{\mathbf{h}_*}\mathcal{A}(X_\delta) \quad (2.21)$$

The index δ in our final norms (2.20) and (2.21) indicate that the large set norm is being taken over polymers in $(\delta\mathbb{Z})^3$. Under each of above norms we have Banach spaces. Moreover it is easy to verify that the *multiplicative* (*Banach algebra*) *property* holds for the polymer activities $\tilde{K}(X_\delta)$ under the $\mathbf{h}$-norm (2.17), the kernel norm (2.18), and, for activities supported on disjoint polymers, under the $\mathbf{h}$, $\mathbf{G}_\kappa$ norm. The multiplicative property plays a very important role in the estimates in the rest of the paper. We therefore state it as Proposition 2.1 below and supply a proof.

Proposition 2.1 *Let $X_{\delta,1}$, $X_{\delta,2}$ denote two connected polymers. Let $X_{\delta,1} = X_{\delta,2}$ or $X_{\delta,1} \cap X_{\delta,2} = \emptyset$. $K_j(X_{\delta,j}, \varphi, \psi)$, $j = 1, 2$ are polymer activities of degree* 0. *Define a new polymer activity*

$$\mathbf{K}(X_{\delta,1} \cup X_{\delta,2}, \varphi, \psi) = K_1(X_{\delta,1}, \varphi, \psi) K_2(X_{\delta,2}, \varphi, \psi)$$

Then

$$\|\mathbf{K}(X_{\delta,1} \cup X_{\delta,2}, \varphi, 0)\|_{\mathbf{h}} \le \|K_1(X_{\delta,1}, \varphi, 0)\|_{\mathbf{h}} \|K_2(X_{\delta,2}, \varphi, 0)\|_{\mathbf{h}}$$

The same inequality holds for the $\mathbf{h}_*$ norm. If $X_{\delta,1}$ and $X_{\delta,2}$ are disjoint we have

$$\|\mathbf{K}(X_{\delta,1} \cup X_{\delta,2})\|_{\mathbf{h},\mathbf{G}_\kappa} \le \|K_1(X_{\delta,1})\|_{\mathbf{h},\mathbf{G}_\kappa} \|K_2(X_{\delta,2})\|_{\mathbf{h},\mathbf{G}_\kappa}$$

Proof Let f_j, $j = 1, \dots, m$ be functions on $\tilde{X}^{(2)}_{\delta,1} \cup \tilde{X}^{(2)}_{\delta,2}$ and $g_{2p}(\mathbf{x}, \mathbf{y}) = g_{2p}(x_1, \dots, x_p, y_1, \dots, y_p)$ be a function on $(\tilde{X}^{(2)}_{\delta,1} \cup \tilde{X}^{(2)}_{\delta,2})^p \times (\tilde{X}^{(2)}_{\delta,1} \cup \tilde{X}^{(2)}_{\delta,2})^p = (\tilde{X}^{(2)}_{\delta,1} \cup \tilde{X}^{(2)}_{\delta,2})^{2p}$. g_{2p} is anti-symmetric in the x_j and in the y_j. By definition

$$\|D^{2p,m}\mathbf{K}(X_{\delta,1} \cup X_{\delta,2}, \varphi, 0)\| = \sup_{\substack{\|f_j\|_{C^2(X_{\delta,1} \cup X_{\delta,2})} \le 1 \\ \|g_{2p}\|_{C^2((X_{\delta,1} \cup X_{\delta,2})^{2p})} \le 1}} |D^{2p,m}\mathbf{K}(X_{\delta,1} \cup X_{\delta,2}, \varphi, 0; f^{\times m}, g_{2p})|$$

where $f^{\times m} = (f_1, \dots, f_m)$, $f^{\times M} = \{f_i\}_{i \in M}$ and $M \subset \{1, 2, \dots, m\}$. We extend the coefficients of $K_j(X_{\delta,j}, \varphi, \psi)$, $j = 1, 2$ to $X_{\delta,1} \cup X_{\delta,2}$ by declaring that they have support in $X_{\delta,j}$. Now D^{2p}_F is a partial (anti) derivation of order $2p$. D^{2m}_B a derivation of order m. Distributing D^{2p}_F and D^{2m}_B on the product of polymer activities gives

$$\begin{aligned}
&D^m_B D^{2p}_F \mathbf{K}(X_{\delta,1} \cup X_{\delta,2}, \varphi, x_1, \dots, x_p, y_1, \dots, y_p); f^{\times m}) \\
&\quad = \sum_{p_1+p_2=p} \sum_{m_1+m_2=m} \sum_{\substack{M_1 \cup M_2 = \{1,\dots,m\} \\ M_1 \cap M_2 = \emptyset \\ |M_1| = m_1, |M_2| = m_2}} \sum_{\substack{I,J \subset \{1,\dots,p\} \\ |I| = |J| = p_1}} D^{m_1}_B D^{2p_1}_F K_1(X_{\delta,1}, \varphi, \mathbf{x}_I, \mathbf{y}_J; f^{\times M_1}) \\
&\qquad \times D^{m_2}_B D^{2p_2}_F K_2(X_{\delta,2}, \varphi, \mathbf{x}_{I^c}, \mathbf{y}_{J^c}; f^{\times M_2}) \\
&\qquad \times g_{2p}(\mathbf{x}_I, \mathbf{x}_{I^c}, \mathbf{y}_J, \mathbf{y}_{J^c}) \times (-1)^\sharp
\end{aligned} \tag{2.22}$$

where $(-1)^\sharp$ is a sign factor which plays no role in the norm bounds to follow, I^c, J^c are respectively the complements of I, J in $\{1, \dots, p\}$. We have $|I^c| = |J^c| = p_2$. We now integrate this with respect to $x_1, \dots, x_p, y_1, \dots, y_p$ in $(X_{\delta,1} \cup X_{\delta,2})^p \times (X_{\delta,1} \cup X_{\delta,2})^p$. Because of the support properties of the coefficients the integral splits over the products on the right hand side. We get

$$
\begin{aligned}
&D^{2p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0;\,f^{\times m},g_{2p})\\
&\quad=\sum_{p_1+p_2=p}\sum_{m_1+m_2=m}\sum_{\substack{M_1\cup M_2=\{1,\dots,m\}\\ M_1\cap M_2=\emptyset\\ |M_1|=m_1,|M_2|=m_2}}\sum_{\substack{I,J\subset\{1,\dots,p\}\\ |I|=|J|=p_1}}\int_{X_{\delta,1}^{2p_1}}d\mathbf{x}_I d\mathbf{y}_J\\
&\qquad\times\int_{X_{\delta,2}^{2p_2}}d\mathbf{x}_{I^c}d\mathbf{y}_{J^c}D_B^{m_1}D_F^{2p_1}K_1(X_{\delta,1},\varphi,\mathbf{x}_I,\mathbf{y}_J;\,f^{\times M_1})\\
&\qquad\times D_B^{m_2}D_F^{2p_2}K_2(X_{\delta,2},\varphi,\mathbf{x}_{I^c},\mathbf{y}_{J^c};\,f^{\times M_2})\\
&\qquad\times g_{2p}(\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c})\times(-1)^{\sharp}
\end{aligned}
\tag{2.23}
$$

where $X_{\delta,j}^{2p_j}=X_{\delta,j}^{p_j}\times X_{\delta,j}^{p_j}$. Define

$$
\begin{aligned}
\tilde g_{2p_1}(\mathbf{x}_I,\mathbf{y}_J)&=\int_{X_{\delta,2}^{2p_2}}d\mathbf{x}_{I^c}d\mathbf{y}_{J^c}D_B^{m_2}D_F^{2p_2}K_2(X_{\delta,2},\varphi,\mathbf{x}_{I^c},\mathbf{y}_{J^c};\,f^{\times M_2})g_{2p}(\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c})\\
&=D^{2p_2,m_2}K_2(X_{\delta,2},\varphi,0;\,g_{2p}(\mathbf{x}_I,\cdot,\mathbf{y}_J,\cdot),f^{\times M_2})
\end{aligned}
\tag{2.24}
$$

where the dependence of $\tilde g_{2p_1}(\mathbf{x}_I,\mathbf{y}_J)$ on K_2, $X_{\delta,2}$, p_2, $f^{\times M_2}$ has been suppressed. Note that $\tilde g_{2p_1}(\mathbf{x}_I,\mathbf{y}_J)$ is antisymmetric in the $\{x_i : i\in I\}$ and in the $\{y_j : j\in J\}$ and therefore qualifies as a test function.

From (2.22) and (2.24) we have

$$
\begin{aligned}
&D^{2p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0;\,f^{\times m},g_{2p})\\
&\quad=\sum_{p_1+p_2=p}\sum_{m_1+m_2=m}\sum_{\substack{M_1\cup M_2=\{1,\dots,m\}\\ M_1\cap M_2=\emptyset\\ |M_1|=m_1,|M_2|=m_2}}\sum_{\substack{I,J\subset\{1,\dots,p\}\\ |I|=|J|=p_1}}D^{2p_1,m_1}K_1\\
&\qquad\times(X_{1,\delta},\varphi,0;\,f^{M_1},\tilde g_{2p_1})\times(-1)^{\sharp}
\end{aligned}
\tag{2.25}
$$

Therefore

$$
\begin{aligned}
&|D^{2p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0;\,f^{\times m},g_{2p})|\\
&\quad\le\sum_{p_1+p_2=p}\sum_{m_1+m_2=m}\sum_{\substack{M_1\cup M_2=\{1,\dots,m\}\\ M_1\cap M_2=\emptyset\\ |M_1|=m_1,|M_2|=m_2}}\sum_{\substack{I,J\subset\{1,\dots,p\}\\ |I|=|J|=p_1}}\|D^{2p_1,m_1}K_1(X_{1,\delta},\varphi,0)\|\\
&\qquad\times\prod_{i\in M_1}\|f_i\|_{C^2(X_{1,\delta})}\|\tilde g_{2p_1}\|_{C^2(X_{1,\delta}^{2p_1})}
\end{aligned}
\tag{2.26}
$$

From (2.24) we have for $0\le k\le 2$

$$
\partial_\delta^k\tilde g_{2p_1}(\mathbf{x}_I,\mathbf{y}_J)=D^{2p_2,m_2}K_2(X_{\delta,2},\varphi,0;\,\partial_\delta^k g_{2p}(\mathbf{x}_I,\cdot,\mathbf{y}_J,\cdot),f^{\times M_2})
$$

where ∂_δ^k is the lattice partial derivative of degree k with respect to $\mathbf{x}_I,\mathbf{y}_J$ in multi-index notation. Whence

$$
|\partial_\delta^k\tilde g_{2p_1}(\mathbf{x}_I,\mathbf{y}_J)|\le\|D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0)\|\prod_{j\in M_2}\|f_j\|_{C^2(X_{\delta,2})}\|\partial_\delta^k g_p(\mathbf{x}_I,\cdot,\mathbf{y}_J,\cdot)\|_{C^2(X_{\delta,2}^{2p_2})}
$$

and therefore

$$\|\tilde{g}_{2p_1}\|_{C^2(X_{\delta,1}^{2p_1})} \leq \|D^{p_2,m_2}K_2(X_{\delta,2},\varphi,0)\| \prod_{j\in M_2} \|f_j\|_{C^2(X_{\delta,2})}\|g_{2p}\|_{C^2(X_{\delta,2}^{2p_2}\times X_{\delta,1}^{2p_1})} \tag{2.27}$$

Now $X_{\delta,2}^{2p_2} \times X_{\delta,1}^{2p_1} \subset (X_{\delta,2}\cup X_{\delta,1})^{2p}$ where $p = p_1 + p_2$. Therefore from (2.26) and (2.27) we get

$$\begin{aligned}
&\|D^{2p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2})\| \\
&\quad \leq \sum_{p_1+p_2=p}\sum_{m_1+m_2=m}\sum_{\substack{M_1\cup M_2=\{1,\dots,m\}\\ M_1\cap M_2=\emptyset\\ |M_1|=m_1,|M_2|=m_2}}\sum_{\substack{I,J\subset\{1,\dots,p\}\\ |I|=|J|=p_1}} \|D^{2p_1,m_1}K_1(X_{1,\delta},\varphi,0)\| \\
&\qquad \times \|D^{2p_2,m_2}K_2(X_{2,\delta},\varphi,0)\|
\end{aligned}$$

Now

$$\sum_{\substack{M_1\cup M_2=\{1,\dots,m\}\\ M_1\cap M_2=\emptyset\\ |M_1|=m_1,|M_2|=m_2}}\sum_{\substack{I,J\subset\{1,\dots,p\}\\ |I|=|J|=p_1}} 1 = \frac{m!}{m_1!m_2!}\frac{(p!)^2}{(p_1!)^2(p_2!)^2}$$

Therefore

$$\begin{aligned}
&\|D^{2p,m}\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0)\| \\
&\quad \leq \sum_{p_1+p_2=p}\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\frac{(p!)^2}{(p_1!)^2(p_2!)^2}\|D^{2p_1,m_1}K_1(X_{1,\delta},\varphi,0)\| \\
&\qquad \times \|D^{2p_2,m_2}K_2(X_{2,\delta},\varphi,0)\|
\end{aligned} \tag{2.28}$$

Multiply both sides of the previous inequality by $h_B^m/m!$ and $h_F^{2p}/(p!)^2$. Sum over integers m, $0\leq m\leq m_0$, and over all integers $p\geq 0$ to obtain

$$\|\mathbf{K}(X_{\delta,1}\cup X_{\delta,2},\varphi,0)\|_{\mathbf{h}} \leq \|K_1(X_{\delta,1},\varphi,0)\|_{\mathbf{h}}\|K_2(X_{\delta,2},\varphi,0)\|_{\mathbf{h}}$$

This proves the first inequality of Proposition 2.1. The second inequality follows from the first because for union of disjoint sets

$$G_\kappa(X_\delta\cup Y_\delta,\varphi) = G_\kappa(X_\delta,\varphi)G_\kappa(Y_\delta,\varphi) \qquad \square$$

3 The RG Map

In this section we describe the RG map applied to the generic density in the polymer representation given in (1.84). This is a lattice transcription of the continuum RG map described in [13] (see also [12]). This goes in several steps. First we must perform the fluctuation integration and rescaling (see (1.72))

$$\mathcal{Z}'(L^{-1}\Lambda_{L^{-1}\delta},\varphi) = S_L\mu_\Gamma * \mathcal{Z}(\Lambda_\delta,\Phi) \tag{3.1}$$

where $\Lambda_\delta\subset(\delta\mathbb{Z})^3$ is the volume arrived at after a certain number of previous RG steps and Γ is the fluctuation covariance for the next step. Γ is one of the covariances Γ_n of

Theorem 1.1 and has the finite range property stated in that theorem. Thus after n RG steps (see (1.68)–(1.73)) $\delta=\delta_n$, $\Gamma=\Gamma_n$, $\Lambda_\delta=\Lambda_{N-n,n}$ and $L^{-1}\Lambda_{L^{-1}\delta}=\Lambda_{N-n-1,n+1}$.

The polymer representation (1.84) for $\mathcal{Z}(\Lambda_\delta)$ is parametrized by the *coordinates* (V,K) on the scale δ where V is a local functional (potential):

$$V(X_\delta)=\sum_{\Delta_\delta\subset X_\delta} V(\Delta_\delta) \tag{3.2}$$

Let $\tilde V(X_\delta,\Phi)$ be an arbitrary local supersymmetric functional with $\tilde V(X_\delta,0)=0$. We will see that the polymer representation is preserved under the RG transformation (3.1) with new coordinates $\tilde V_L,\mathcal{F}(K)$ on the next scale $L^{-1}\delta$. $\mathcal{F}$ depends on $\tilde V$. The finite range property of Γ leads to a simple description of this map:

$$\begin{aligned}
&V\to\tilde V_L,\quad \tilde V_L(\Delta_{L^{-1}\delta},\Phi)=(S_LV)(\Delta_{L^{-1}\delta},\Phi)=\tilde V(L\Delta_\delta,S_L\Phi)\\
&K\to\mathcal{F}(K),\quad \mathcal{F}(K)(X_{L^{-1}\delta},\Phi)=\int d\mu_\Gamma(\xi)\mathcal{B}K(LX_\delta,\xi,S_L\Phi)
\end{aligned}\tag{3.3}$$

where $\mathcal{B}K$ is a $\tilde V$ dependent nonlinear functional of K to be presently described. We call this map the *fluctuation map*.

We can take advantage of the arbitrariness of the local potential $\tilde V$ in the above map so as to remove the expanding (relevant) parts F in the polymer activity $\mathcal{F}(K)$ and compensate by a change $\tilde V_L(F)$ in the local potential $\tilde V_L$ in such a way that the evolved density $\mathcal{Z}'(L^{-1}\Lambda_{L^{-1}\delta})$ on the left hand side of (3.1) remains unchanged. This operation gives rise to the *extraction* map [6]

$$\tilde V_L\to V'(F)=\tilde V_L-\tilde V_L(F),\qquad \mathcal{F}(K)\to K'=\mathcal{E}(\mathcal{F}(K),F) \tag{3.4}$$

where the image is on the same scale $L^{-1}\delta$. $V'(F)$ and the nonlinear map $\mathcal{E}$ have simple expressions which are lattice transcriptions of those given in [6]. The composition of the fluctation map (3.3) and the extraction map (3.4) gives the *RG map*

$$f:\ f(V,K)=(f_V(V,K),f_K(V,K))$$

where

$$\begin{aligned}
&f_V: V\to\tilde V_L\to V'(F)\\
&f_K: K\to\mathcal{F}(K)\to K'=\mathcal{E}(\mathcal{F}(K),F)
\end{aligned}\tag{3.5}$$

The operation of extraction leads in particular to a discrete flow of the coupling constants in V on scale $L^{-1}\delta$ provided we choose F, $\tilde V_L(F)$ appropriately. The expanding functionals will be gathered in the local potential $V'(F)$ whereas the polymer activity $\mathcal{E}(\mathcal{F}(K),F)$ will be a contracting (irrelevant) error term.

3.1 The Fluctuation Map

We now construct the map (3.3) starting from (3.1) with the density in the polymer representation (1.84). In performing the fluctuation integration

$$\mu_\Gamma*\mathcal{Z}(\Lambda_\delta,\Phi)=\int d\mu_\Gamma(\xi)\sum_N\frac{1}{N!}e^{-V(X_\delta^{(c)},\Phi+\xi)}\sum_{X_{\delta,1},\dots,X_{\delta,N}}\prod_{j=1}^N K(X_{\delta,j},\Phi+\xi) \tag{3.6}$$

we will exploit the independence of $\xi(x)$ and $\xi(y)$ when $|x-y|\geq L$. To this end we construct an L-paving of Λ_δ and the L-closure of $\bar{X_\delta}^{(L)}$ of a connected 1-polymer X_δ as in the paragraph preceding Lemma 2.1. The 1-polymers will be combined into larger connected L-polymers which by definition are connected unions of L-blocks (for the relevant definitions intervening here and in the following see the paragraph on *L-polymers and L-closures* before Lemma 2.1). The combination is performed in such a way that the new polymers are associated to independent functionals of ξ. This is the lattice adaptation of Sect. 3.1 of [13].

Define the polymer activity P, supported on unit blocks, by:

$$P(\Delta_\delta,\xi,\Phi)=e^{-V(\Delta_\delta,\xi+\Phi)}-e^{-\tilde{V}(\Delta_\delta,\Phi)} \tag{3.7}$$

with $\tilde{V}$, to be chosen. $\tilde{V}(\Delta_\delta,\Phi)$ is required to satisfy $\tilde{V}(\Delta_\delta,0)=0$. *In the following V, K has field argument $\xi+\Phi$ whereas $\tilde{V}$ depends only on Φ*. The dependence of P on ξ,Φ is as defined above. $X_\delta^c=\Lambda_\delta\setminus\bigcup_{j=1}^N X_{\delta,j}$ is a union of disjoint 1-blocks Δ_δ. Therefore

$$e^{-V(X_\delta^c)}=\prod_{\Delta_\delta\subset X_\delta^c}[e^{-\tilde{V}(\Delta_\delta)}+P(\Delta_\delta)]$$

Expand the product and insert the expansion into the integrand of in (3.6) which gives

$$\text{integrand}=\sum_N\frac{1}{N!}\sum_{(X_{\delta,j}),(\Delta_{\delta,i})}e^{-\tilde{V}(X_{\delta,0})}\prod_{j=1}^N K(X_{\delta,j})\prod_{i=1}^M P(\Delta_{\delta,i}) \tag{3.8}$$

where $X_{\delta,0}=\Lambda_\delta\setminus(\bigcup X_{\delta,j})\cup(\bigcup\Delta_{\delta,i})$. Let Y_δ be the L-closure of $(\bigcup X_{\delta,j})\cup(\bigcup\Delta_{\delta,i})$ and let $Y_{\delta,1},\dots,Y_{\delta,P}$ be the connected components of Y_δ. These are L-polymers. Let f be the function that maps $\pi:=(X_{\delta,j}),(\Delta_{\delta,i})$ into $\{Y_{\delta,1},\dots,Y_{\delta,P}\}$. Now we perform the sum over $(X_{\delta,j}),(\Delta_{\delta,i})$ in (3.8) by summing over $\pi\in f^{-1}(\{Y_{\delta,1},\dots,Y_{\delta,P}\})$ and then $\{Y_{\delta,1},\dots,Y_{\delta,P}\}$. The result is:

$$\text{integrand}=\sum_N\frac{1}{N!}\sum_{(Y_{\delta,j})}e^{-\tilde{V}(Y_\delta^c)}\prod_{j=1}^N\mathcal{B}K(Y_{\delta,j}) \tag{3.9}$$

where the sum is over strictly disjoint connected L polymers and

$$(\mathcal{B}K)(Y_\delta)=\sum_{N+M\geq 1}\frac{1}{N!M!}\sum_{(X_{\delta,j}),(\Delta_{\delta,i})\to\{Y\}}e^{-\tilde{V}(X_{\delta,0})}\prod_{j=1}^N K(X_{\delta,j})\prod_{i=1}^M P(\Delta_{\delta,i}) \tag{3.10}$$

where $X_{\delta,0}=Y\setminus(\bigcup X_{\delta,j})\cup(\bigcup\Delta_{\delta,i})$ and the $\to$ is the map f. In other words the sum in (3.10) is over distinct $\Delta_{\delta,i}$ and disjoint 1-polymers $X_{\delta,j}$ such that their L-closure is the connected L-polymer Y_δ.

We now perform the fluctuation integration of (3.9) over ξ followed by rescaling. Since $\tilde{V}(Y_\delta^c)$ is independent of ξ the ξ integration factors through and acts on the product of polymer activities $\prod_j(\mathcal{B}K)(Y_{\delta,j})$. A polymer activity $(\mathcal{B}K)(Y_{\delta,j})$ belongs to $\Omega^0(\tilde{Y}_{\delta,j}^{(2)})$. The $Y_{\delta,j}$ are strictly disjoint connected L-polymers and thus necessarily separated from each other by a distance $\geq L$. The 2-collar attached L-polymers $\Omega^0(\tilde{Y}_{\delta,j}^{(2)})$ are therefore separated from each other by a distance $\geq L-4$. The fluctuation covariance Γ has finite range $L/2$ and for L sufficiently large $L-4\geq L/2$. Therefore the fluctuation integration over the product of polymer activities factorizes. We now follow this up by applying the rescaling operator to both sides. This has the effect of bringing us back to 1-polymers but on the scale $L^{-1}\delta$. Therefore we obtain

$$\begin{aligned}&(S_L\mu_\Gamma * \mathcal{Z})(L^{-1}\Lambda_{L^{-1}\delta}, \Phi)\\&\quad = \sum_N \frac{1}{N!} \sum_{(X_{L^{-1}\delta,j})} e^{-\tilde{V}_L(X^c_{L^{-1}\delta}, \Phi)} \prod_{j=1}^{N} \int d\mu_\Gamma(\xi) \mathcal{B}K(LX_{\delta,j}, S_L\Phi, \xi)\end{aligned} \tag{3.11}$$

where $X_{L^{-1}\delta,j} = X_j \cap (L^{-1}\delta\mathbb{Z})^3$ (as well as $X_{\delta,j} = X_j \cap (\delta\mathbb{Z})^3$) are disjoint 1-polymers, $X^c_{L^{-1}\delta} = L^{-1}\Lambda_{L^{-1}\delta} \setminus \bigcup_j X_{L^{-1}\delta,j}$ and $\tilde{V}_L(\Delta_{L^{-1}\delta}) = S_L\tilde{V}(L\Delta_\delta)$. This gives the fluctuation map (3.3): $V \to \tilde{V}_L$, $K \to \mathcal{F}(K)$ with $\mathcal{B}K$ defined as above. At the same time we have shown that the polymer representation is stable with respect to the RG transformation.

Consider

$$\mathcal{F}(K)(X_{L^{-1}\delta}, \Phi) = \int d\mu_\Gamma(\xi) \mathcal{B}K(LX_\delta, \xi, S_L\Phi)$$

By construction $\mathcal{B}K$ is supersymmetric. Therefore since the supersymmetry operator commutes with the measure $\mathcal{F}(K)$ is also supersymmetric. Now since $P(\Delta_\delta, \xi, 0)$ and $K(X_\delta, \xi)$ vanish for $\xi = 0$ (the latter by hypothesis, see (1.87)) it follows that $\mathcal{B}K(LX_\delta, \xi, 0)$ also vanishes for $\xi = 0$. Therefore by Lemma 1.1

$$\mathcal{F}(K)(X_{L^{-1}\delta}, 0) = \int d\mu_\Gamma(\xi) \mathcal{B}K(LX_\delta, \xi, 0) = \mathcal{B}K(LX_\delta, 0, 0) = 0 \tag{3.12}$$

Thus the condition (1.87) is satisfied by the new polymer activities. This implies in particular that no field independent relevant parts are generated by the fluctuation integration as a consequence of supersymmetry.

3.2 Extraction

Let $\delta' = L^{-1}\delta$ and let $\Lambda' = L^{-1}\Lambda_{\delta'}$. The fluctuation map gave us $\tilde{V}_L, \mathcal{F}(K)$ as the coordinates of the evolved density $\mathcal{Z}(\Lambda')$. We want to change the local potential $\tilde{V}_L$ and the polymer activity $\mathcal{F}(K)$ simultaneously such that $\mathcal{Z}(\Lambda')$ remains invariant. To this end let $P(\Phi(x))$ be a *local* polynomial, which means that it is a polynomial in $\Phi(x)$ for $x \in \Lambda'$. Furthermore we require that $P(0) = 0$, i.e. P has no field independent part.

Given $\Delta_{\delta'}$ a unit block in Λ' we consider a change in $\tilde{V}_L(\Delta_{\delta'})$ of the form

$$\tilde{V}_L(F)(\Delta_{\delta'}) = \sum_P \int_{\Delta_{\delta'}} dx\, \alpha_P(x) P(\Phi(x)) \tag{3.13}$$

where the sum ranges over finitely many local polynomials and, for each such P, $\alpha_P(x)$ has the form

$$\alpha_P(x) = \sum_{X_{\delta'} \supset x} \alpha_P(X_{\delta'}, x) \tag{3.14}$$

such that $\alpha_P(X_{\delta'}, x) = 0$ if $x \notin X_{\delta'}$, $\alpha_P(X_{\delta'}, x) = 0$ if $X_{\delta'} \not\subset \Lambda'$ and $\alpha_P(X_{\delta'}, x) = 0$ if $X_{\delta'}$ is not a small set (see definition after (2.8)). The corresponding change in $\mathcal{F}(K)$ is given in terms of the *relevant parts*

$$\begin{aligned}F(X_{\delta'}, \Phi) &= \sum_P \int_{X_{\delta'}} dx\, \alpha_P(X_{\delta'}, x) P(\Phi(x)),\\ F(X_{\delta'}, \Delta_{\delta'}) &= \sum_P \int_{\Delta_{\delta'}} dx\, \alpha_P(X_{\delta'}, x) P(\Phi(x))\end{aligned} \tag{3.15}$$

Note that $F(X_{\delta'}, 0) = 0$.

Extraction Map

Theorem 3.1 (After Brydges, Dimock and Hurd [6]) *Given $F,\ \tilde V_L(F)$ as above there exists a polymer activity which is a non-linear functional $\mathcal{E}(\mathcal{F}(K),F)$ of $\mathcal{F}(K),\ F$ such that*

$$L\to V'(F)=\tilde V_L-\tilde V_L(F),\qquad \mathcal{F}(K)\to K'=\mathcal{E}(\mathcal{F}(K),F) \tag{3.16}$$

preserves the polymer representation for the density $\mathcal{Z}(\Lambda')$ with new coordinates $(V',\ K')$ satisfying $V'(F)(\Delta_{\delta'},0)=K'(X_{\delta'},0)=0$. Let $\mathcal{E}_1$ denote the linearization of $\mathcal{E}$. Then the linearization of the extraction map is given by

$$\mathcal{E}_1(\mathcal{F}(K),F)=\mathcal{F}(K)-Fe^{-\tilde V_L},\qquad V'(F)=\tilde V_L-\tilde V_L(F) \tag{3.17}$$

We say that $\tilde V_L$ is stable with respect to perturbation F if there are positive numbers $f(X)$ such that

$$\left\|e^{-\tilde V_L(\Delta_{\delta'})-\sum_{X_{\delta'}\supset\Delta_{\delta'}}z(X)F(X_{\delta'},\Delta_{\delta'})}\right\|_{\mathbf{h},G_\kappa}\le 2 \tag{3.18}$$

for all complex numbers $z(X_{\delta'})$ with $|z(X_{\delta'})|f(X_{\delta'})\le 2$. Assume that $\tilde V_L$ is stable. Then $\mathcal{E}(\mathcal{F}(K),F)$ is norm analytic and satisfies the bounds

$$\|\mathcal{E}(\mathcal{F}(K),F)\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta'}\le O(1)(\|\mathcal{F}(K)\|_{\mathbf{h},G_\kappa,\mathcal{A}_1,\delta'}+\|f\|_{\mathcal{A}_3,\delta'}) \tag{3.19}$$

$$|\mathcal{E}(\mathcal{F}(K),F)|_{\mathbf{h},\mathcal{A},\delta'}\le O(1)(\|\mathcal{F}(K)\|_{\mathbf{h},\mathcal{A}_1,\delta'}+\|f\|_{\mathcal{A}_3,\delta'}) \tag{3.20}$$

Proof This is a restatement of Theorem 5 in Sect. 4.2 of [6] with the substitution $(\tilde V_L,\mathcal{F}(K))$ for (V,K), adapted to the lattice. The proof of Theorem 5 exploited Lemmas 10, 11, 12, 13 the last of them providing the extraction formula in (121), p. 781 of [6]. In [6] the continuum unit blocks are open. Our lattice unit blocks are lattice restrictions of continuum open unit cubes. Overlap connectedness is replaced by connectedness. With this in mind the proofs of Lemmas 10, 11, 12, 13 go through intact on the lattice providing the extraction map above. The estimates in Theorem 5 on the norms of $\mathcal{E}(K,F)$ together with norm analyticity remain valid on the lattice. □

Remark The stability property (3.18) is proved in Sect. 5 once we have chosen $\tilde V$ appropriately. The estimate (3.19) on the extraction operator $\mathcal{E}$ plays an essential role and is exploited in Sect. 5.

Formal Infinite Volume Limit We reestablish the notations leading to (1.73). Choose $\delta=\delta_n,\ \Gamma=\Gamma_n,\ \delta'=L^{-1}\delta=\delta_{n+1}$. $\Lambda_\delta=\Lambda_{N-n,n},\ \Lambda_{\delta'}=\Lambda_{N-n-1,n+1}$ and $\mathcal{F}=\mathcal{F}_{n+1}$ in (3.3). The RG transformation $T_{N-n-1,n+1}$ of (1.73) induces the RG map $f_{N-n-1,n+1}(V,K)$ of (3.5) for the coordinates of the density $\mathcal{Z}_{n-1}(\Lambda_{N-n,n})$ in the polymer representation. $\alpha_P(X_{\delta_{n+1}},x)$ in (3.14) is chosen later in Sect. 4. This choice will be local, in the sense that it is determined by $\tilde V_L(\Delta_{\delta_{n+1}}),\ \Delta_{\delta_{n+1}}\subset X_{\delta_{n+1}}$ and by $\mathcal{F}_{n+1}(K)(X_{\delta_{n+1}})$. Lemma 13 and (112) of [6] imply that $\mathcal{E}(\mathcal{F}_{n+1}(K),F))(X_{\delta_{n+1}})$ also is local: it is determined by $\mathcal{F}_{n+1}(K)(Y_{\delta_{n+1}}),\ Y_{\delta_{n+1}}\subset X_{\delta_{n+1}}$ and $\tilde V_L(\Delta_{\delta_{n+1}}),\ \Delta_{\delta_{n+1}}\subset\tilde X_{\delta_{n+1}}$, where $\tilde X_{\delta_{n+1}}$ is a neighbourhood of $X_{\delta_{n+1}}$, namely the union of $X_{\delta_{n+1}}$ with all small sets that intersect X_{δ_n}. Therefore the K component of the map $f_{N-n-1,n+1}$ representing the action of the $n+1$th step of RG, namely $f_{N-n-1,n+1,K}(K,V)(X_{\delta_{n+1}},\Phi)$ is independent of N for all N large enough so that $\Lambda_{N-n-1,n+1}$ contains $\tilde X_{\delta_{n+1}}$. Thus $\lim_{N\to\infty}f_{N-n-1,n+1,K}(K,V)(X_{\delta_{n+1}},\Phi)$ exists pointwise in $X_{\delta_{n+1}}$. In this paper we are studying the action of this pointwise infinite volume limit called the *formal infinite volume limit*.

3.3 Appendix

We record here some definitions which have either already been used or will be used later. The object is to be able to move scaling past fluctuation integration.

We define for $x, y \in (L^{-1}\delta\mathbb{Z})^3$ and any covariance u on $(\delta\mathbb{Z})^3$

$$u_L(x-y) = S_{L^{-1}}u(x-y) = L^{2d_s}u(L(x-y)) \tag{3.21}$$

Since the fluctuation covariance Γ defined on $(\delta\mathbb{Z})^3$ has finite range $L/2$ we have that Γ_L defined on $(L^{-1}\delta\mathbb{Z})^3$ has finite range $1/2$. We recall from Sect. 1.3 that a polymer X_δ is defined by $X_\delta = X \cap (\delta\mathbb{Z})^3$ where X is a continuum polymer. We define the rescaling of polymer activities by

$$S_L K(X_{L^{-1}\delta}, \Phi) = K_L(X_{L^{-1}\delta}, \Phi) = K(LX_\delta, S_L\Phi) \tag{3.22}$$

We write the fluctuation integration of the polymer activity $K(X_\delta, \Phi, \xi)$ with respect to μ_Γ as

$$K^\sharp(X_\delta, \Phi) = \int d\mu_\Gamma(\xi) K(X_\delta, \Phi, \xi) \tag{3.23}$$

We write the fluctuation integration of the polymer activity $K(X_{L^{-1}\delta}, \Phi, \xi)$ with respect to μ_{Γ_L} as

$$K^\natural(X_{L^{-1}\delta}, \Phi) = \int d\mu_{\Gamma_L}(\xi) K(X_{L^{-1}\delta}, \Phi, \xi) \tag{3.24}$$

We define

$$\mathcal{S}_L = S_L\mathcal{B} \tag{3.25}$$

With these notations it is easy to see that the fluctuation map can be written as

$$\mathcal{F}(K)(X_{L^{-1}\delta}, \Phi) = (\mathcal{B}K)^\sharp(LX_\delta, S_L\Phi) = (\mathcal{S}_L K)^\natural(X_{L^{-1}\delta}, \Phi) \tag{3.26}$$

4 The Renormalization Group Map Applied

In this section we specify the RG map of Sect. 3 by making choices for the local potential $\tilde{V}$, and relevant parts F. $\tilde{V}$ is chosen via first order perturbation theory. F is chosen so as to remove the expanding part of the fluctuation map. This is the extraction step. This will be done in second order perturbation theory as well as in the error term. We will follow closely the strategy in Sect. 4 of [13]. We will use the notations established in the Appendix to Sect. 3.3, (3.21)–(3.26). We take $\delta = \delta_n$, $\Gamma = \Gamma_n$. Recall that, see (1.61), $\alpha = \frac{3+\varepsilon}{2}$ where we take $0 < \varepsilon < 1$. The field scaling dimension is $d_s = \frac{3-\varepsilon}{4}$, see (1.62), (1.63).

We assume that starting from the unit lattice where only the local potential (1.35) is present n steps of the renormalization group map has been carried out. This produces a new local potential

$$
\begin{aligned}
V_n(\Delta_{\delta_n},\Phi) &= V(\Delta_{\delta_n},\Phi,C_n,g_n,\mu_n)\\
&= g_n\int_{\Delta_{\delta_n}} dx{:}(\Phi\bar\Phi)^2(x){:}_{C_n} + \mu_n\int_{\Delta_{\delta_n}} dx(\Phi\bar\Phi)(x)
\end{aligned}
\tag{4.1}
$$

together with a polymer activity K_n supported on connected polymers in $(\delta_n\mathbb{Z})^3$. Note that $(\Phi\bar\Phi)(x) = {:}\Phi\bar\Phi{:}_{C_n}(x)$ by virtue of (1.31). We write K_n in the form

$$
K_n = Q_n e^{-V_n} + R_n \tag{4.2}
$$

where Q_n is a polymer activity which is given by second order perturbation theory in g assuming that μ is $O(g^2)$. Q_n is specified below. R_n is the remainder which is formally of $O(g^3)$. Q_n, R_n vanish when $\Phi = 0$ by hypothesis. The RG map will preserve this property.

In order to carry through the next step of the RG map as described in Sect. 3 we must also specify $\tilde V(\Delta_{\delta_n},\Phi)$. We define

$$
\begin{aligned}
\tilde V_n(\Delta_{\delta_n},\Phi) &= V(\Delta_{\delta_n},\Phi,C_{n+1,L^{-1}},g_n,\mu_n)\\
&= g_n\int_{\Delta_{\delta_n}} dx{:}(\Phi\bar\Phi)^2(x){:}_{C_{n+1,L^{-1}}} + \mu_n\int_{\Delta_{\delta_n}} d^3x(\Phi\bar\Phi)(x)
\end{aligned}
\tag{4.3}
$$

where we have used the notation $C_{n+1,L^{-1}} = S_L C_{n+1}$. Here and in what follows we adopt the notations introduced in the Appendix of Sect. 3.3. Thus $\sharp$ denotes fluctuation integration with respect to the measure $d\mu_{\Gamma_n}(\xi)$ and $\natural$ denotes fluctuation integration with respect to the measure $d\mu_{\Gamma_{n,L}}(\xi)$, with $\Gamma_{n,L} = S_{L^{-1}}\Gamma_n$. We recall (see Sect. 3.1) that when we perform the fluctuation integration the fluctuation field ξ enters V through $V(\Delta_{\delta_n},\Phi+\xi)$ but $\tilde V$ will remain independent of ξ.

We now define Q_n: Q_n is supported on connected polymers X_{δ_n} such that $|X_{\delta_n}| \le 2$. We assume it can be written in the form

$$
Q_n(X_{\delta_n},\Phi) = Q(X_{\delta_n},\Phi;C_n,\mathbf{w}_n,g_n) = g_n^2\sum_{j=1}^{3} Q^{(j,j)}(\hat X_{\delta_n},\Phi;C_n,w_n^{(4-j)}) \tag{4.4}
$$

where $\mathbf{w}_n = (w_n^{(1)}, w_n^{(2)}, w_n^{(3)})$ is a triple of integral kernels to be obtained inductively and

$$
\hat X_{\delta_n} = \begin{cases} \Delta_{\delta_n}\times\Delta_{\delta_n} & \text{if } X_{\delta_n} = \Delta_{\delta_n}\\ (\Delta_{\delta_n,1}\times\Delta_{\delta_n,2})\cup(\Delta_{\delta_n,2}\times\Delta_{\delta_n,1}) & \text{if } X_{\delta_n} = \Delta_{\delta_n,1}\cup\Delta_{\delta_n,2}\\ 0 & \text{otherwise}\end{cases} \tag{4.5}
$$

$$
\begin{aligned}
Q^{(1,1)}(\hat X_{\delta_n},\Phi;C_n,w_n^{(3)}) &= -2\int_{\hat X_{\delta_n}} dxdy(\Phi(x)-\Phi(y))(\bar\Phi(x)-\bar\Phi(y))w_n^{(3)}(x-y)\\
Q^{(2,2)}(\hat X_{\delta_n},\Phi;C_n,w_n^{(2)}) &= -\int_{\hat X_{\delta_n}} dxdy[{:}(\Phi(x)-\Phi(y))(\bar\Phi(x)-\bar\Phi(y))(\Phi(x)\\
&\qquad +\Phi(y))(\bar\Phi(x)+\bar\Phi(y)){:}_{C_n}\\
&\qquad +3:[(\Phi\bar\Phi)(x)-(\Phi\bar\Phi)(y)]^2:_{C_n}]w_n^{(2)}(x-y)\\
Q^{(3,3)}(\hat X_{\delta_n},\Phi;C_n,w_n^{(1)}) &= 4\int_{\hat X_{\delta_n}} dxdy\ {:}\Phi(x)\bar\Phi(x)\Phi(x)\bar\Phi(y)\Phi(y)\bar\Phi(y){:}_{C_n} w_n^{(1)}(x-y)
\end{aligned}
\tag{4.6}
$$

Note that in the expression for $Q^{(1,1)}$ is equal to its C_n Wick ordered form because of (1.31).

Next we define the second order approximation to the RG map. Let p_n be the activity supported on unit blocks defined by

$$p_n(\Delta_{\delta_n},\xi,\Phi)=V_n(\Delta_{\delta_n},\xi+\Phi)-\tilde{V}_n(\Delta_{\delta_n},\Phi)=p_{n,g}+p_{n,\mu} \tag{4.7}$$

where

$$\begin{aligned} p_{n,g}=g\int_{\Delta_{\delta_n}}dx\Big(&:(\xi\bar{\xi})^2:_{\Gamma_n}(x)+2\sum_{\alpha}\big[\Phi_\alpha(x):\bar{\xi}_\alpha(\xi\bar{\xi}):_{\Gamma_n}(x)+:(\xi\bar{\xi})\xi_\alpha:_{\Gamma_n}(x)\Phi_a(x)\big]\\ &+2(\Phi\bar{\Phi})(x)(\xi\bar{\xi})(x)+(\Phi\bar{\xi})^2(x)+(\xi\bar{\Phi})^2(x)\\ &+2\sum_{\alpha,\beta}:(\xi_\alpha\bar{\xi}_\beta):_{\Gamma_n}(x):(\bar{\Phi}_\alpha\Phi_\beta)(x):_{C_{n+1,L^{-1}}}\\ &+2\sum_{\alpha}\big[\xi_\alpha(x):\bar{\Phi}_\alpha(\Phi\bar{\Phi}):_{C_{n+1,L^{-1}}}(x)+:(\Phi\bar{\Phi})\Phi_\alpha:_{C_{n+1,L^{-1}}}(x)\bar{\xi}_\alpha(x)\big]\Big)\\ p_{n,\mu}=\mu\int_{\Delta_{\delta_n}}&dx\,\big((\xi\bar{\Phi})(x)+(\Phi\bar{\xi})(x)+(\xi\bar{\xi})(x)\big) \end{aligned} \tag{4.8}$$

In (4.8) we have used a component notation. Thus $\Phi_1=\varphi,\ \Phi_2=\psi$. Similarly for the fluctuation field ξ, $\xi_1=\zeta,\ \xi_2=\eta$. ζ is bosonic (degree 0)and η fermionic (degree 1). In deriving (4.8) from (4.7) have used $C_n=\Gamma_n+C_{n+1,L^{-1}}$ (see (1.54)), the independence of $\Phi,\ \xi$ in the sense that their components are independent and distributed with covariances $C_{n+1,L^{-1}},\ \Gamma_n$ respectively. The unordered objects in (4.8) are equal to their Wick ordered form.

We will effectuate the RG map of Sect. 3 following closely the strategy in [13]. Namely, we insert a complex parameter λ into our previous definitions in such a way that (i) at $\lambda=1$ our λ dependent objects correspond with the previous definitions. (ii) The expansion through order λ^2 is second order perturbation theory in g_n counting $\mu_n=O(g_n^2)$. (iii) Powers of λ are determined so as to correspond with leading powers of g_n buried inside polymer activities. (iv) All functions will turn out to be norm analytic in λ and this will enable us in Sect. 5 to profit from Cauchy estimates.

We define

$$P_n(\lambda)=e^{-\tilde{V}_n}\left(-\lambda p_{n,g}-\lambda^2 p_{n,\mu}+\frac{1}{2}\lambda^2 p_{n,g}^2\right)+\lambda^3 r_{n,1} \tag{4.9}$$

where $r_{n,1}$ is defined by the condition $P_n(\lambda=1)=P_n=e^{-V_n}-e^{-\tilde{V}_n}$. Similarly, we define

$$K_n(\lambda)=\lambda^2 e^{-\tilde{V}_n}Q_n+\lambda^3\big([e^{-V_n}-e^{-\tilde{V}_n}]Q_n+R_n\big) \tag{4.10}$$

which, for $\lambda=1$ coincides with $K_n=e^{-V_n}Q_n+R_n$. Corresponding to (3.10) we define

$$\begin{aligned}\mathcal{B}(\lambda,K_n)(Y_{\delta_n})=\sum_{N+M\ge 1}\frac{1}{N!M!}&\sum_{(X_{\delta_n,j}),(\Delta_{\delta_n,i})\to\{Y_{\delta_n}\}}e^{-\tilde{V}_n(X_{\delta_n,0})}\\ &\times\prod_{j=1}^{N}K_n(\lambda,X_{\delta_n,j})\prod_{i=1}^{M}P_n(\lambda,\Delta_{\delta_n,i})\end{aligned} \tag{4.11}$$

where $X_{\delta_n,0}=Y_{\delta_n}\setminus(\bigcup X_{\delta_n,j})\cup(\bigcup\Delta_{\delta_n,i})$. Let $\mathcal{S}(\lambda,K_n)=S_L\mathcal{B}(\lambda,K_n)$, where S_L is the rescaling defined in the last section.

The RG map (see Sect. 3) for K_n with parameter λ is $K_n \mapsto f_{n+1,K}(\lambda, K_n) = \mathcal{E}(\mathcal{S}(\lambda, K_n)^\natural, F_n(\lambda))$, where the superscript $\natural$ denotes integration over the fluctuation field $\xi = (\zeta, \eta)$ with the measure $d\mu_{\Gamma_{n,L}}$ and $\Gamma_{n,L}$ is the rescaled covariance $S_{L^{-1}}\Gamma_n$ as in the Appendix to Sect. 3. The relevant part $F_n(\lambda))$ is defined on polymers in $(\delta_{n+1}\mathbb{Z})^3$ and will be written as

$$F_n(\lambda) = \lambda^2 F_{Q_n} + \lambda^3 F_{R_n} \tag{4.12}$$

and $F_n(\lambda) = F_n$, when $\lambda = 1$.

Perturbative Contribution to f_{n+1}

Given a function $f(\lambda)$ let

$$T_\lambda f = f(0) + f'(0) + \frac{1}{2} f''(0) \tag{4.13}$$

be the Taylor expansion to second order evaluated at $\lambda = 1$. Then the second order approximation to the RG map is $f_{n+1}^{(\leq 2)} = (f_{n+1,K}^{(\leq 2)}, f_{n+1,V}^{(\leq 2)})$ with

$$\begin{aligned} f_{n+1,K}^{(\leq 2)}(K_n, V_n) &= T_\lambda \mathcal{E}(\mathcal{S}(\lambda, K_n)^\natural, F_n(\lambda)) = \mathcal{E}_1(T_\lambda \mathcal{S}(\lambda, K_n)^\natural, F_{Q_n}), \\ f_{n+1,V}^{(\leq 2)}(K_n, V_n) &= V_{n+1}^{(\leq 2)} \end{aligned} \tag{4.14}$$

where

$$V_{n+1}^{(\leq 2)} = \tilde{V}_{n,L} - \tilde{V}_{n,L}(F_{Q_n})$$

Note also that only the linearized $\mathcal{E}_1$ intervenes, because it will turn out that the nonlinear part of extraction generates terms only at order λ^3 or higher.

Proposition 4.1 *There is a choice of F_Q such that the form of Q remains invariant under the RG evolution at second order. In more detail, $f_{n+1}^{(\leq 2)}(V_n, Q_n e^{-V_n}) = (V_{n+1}^{(\leq 2)}, Q_{n+1}^{(\leq 2)} e^{-\tilde{V}_{n,L}})$ where the parameters in*

$$V_{n+1}^{(\leq 2)}(\Delta_{\delta_{n+1}}) = V(\Delta_{\delta_{n+1}}, C_{n+1}, g'_{n+1,(\leq 2)}, \mu'_{n+1,(\leq 2)})$$

evolved according to

$$g_{n+1,(\leq 2)} = L^\varepsilon g_n (1 - L^\varepsilon a_n g_n), \qquad \mu'_{n+1,(\leq 2)} = L^{\frac{3+\varepsilon}{2}} \mu_n - L^{2\varepsilon} b_n g_n^2 \tag{4.15}$$

The parameters in $Q_{n+1}^{(\leq 2)} = Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n,L})$, where $g_{n,L} = L^\varepsilon g_n$, evolved according to

$$\mathbf{w}_{n+1} = \mathbf{v}_{n+1} + \mathbf{w}_{n,L}, \qquad v_{n+1}^{(1)} = \Gamma_{n,L}, \qquad v_{n+1}^{(p)} = (C_{n,L})^p - (C_{n+1})^p, \quad p = 2, 3 \tag{4.16}$$

The constants a_n, b_n are given by

$$a_n = 4 \int_{(\delta_{n+1}\mathbf{Z})^3} dy\, v_{n+1}^{(2)}(y), \qquad b_n = 2 \int_{(\delta_{n+1}\mathbf{Z})^3} dy\, v_{n+1}^{(3)}(y) \tag{4.17}$$

Proof We define a polymer activity $\hat{Q}_{n,L}$ supported on connected polymers $X_{\delta_{n+1}}$ with $|X_{\delta_{n+1}}| \leq 2$ as follows: if $|X_{\delta_{n+1}}| = 1$, say $X_{\delta_{n+1}} = \Delta_{\delta_{n+1}}$, then

$$\hat{Q}_{n,L}(\Delta_{\delta_{n+1}}, \xi, \Phi) = \frac{1}{2}(p_{n,L}(\Delta_{\delta_{n+1}}, \xi, \Phi))^2$$

If $|X_{\delta_{n+1}}| = 2$ then

$$\hat{Q}_{n,L}(X_{\delta_{n+1}}, \xi, \Phi) = \frac{1}{2} \sum_{\substack{\Delta_{n+1,1}, \Delta_{n+1,2} \\ \Delta_{n+1,1} \cup \Delta_{n+1,2} = X_{\delta_{n+1}}}} p_{n,L,g}(\Delta_{n+1,1}, \xi, \Phi) p_{n,L,g}(\Delta_{n+1,2}, \xi, \Phi) \tag{4.18}$$

where $p_{n,L,g}$ is defined by replacing in (4.7) and (4.8) $(g_n, \mu_n, \Gamma_n, C_{n+1,L^{-1}})$ by $(g_{n,L}, \mu_{n,L}, \Gamma_{n,L}, C_{n+1})$ with $g_{n,L} = L^{\varepsilon} g_n$ and $\mu_{n,L} = L^{\frac{3+\varepsilon}{2}} \mu_n$.

It is easy to check that

$$T_\lambda \mathcal{S}(K_n, \lambda) = -p_{n,L} e^{-\tilde{V}_{n,L}} + (e^{-\tilde{V}_{n,L}} \hat{Q}_{n,L} + e^{-\tilde{V}_{n,L}} Q_{n,L}) \tag{4.19}$$

where

$$Q_{n,L}(X_{\delta_{n+1}}, \xi + \Phi) = Q(X_{\delta_{n+1}}, \xi + \Phi, C_{n,L}, \mathbf{w}_{n,L}, g_{n,L})$$

Using $C_{n,L} = \Gamma_{n,L} + C_{n+1}$ and remembering that $\tilde{V}$ depends only on Φ we get

$$(e^{-\tilde{V}_{n,L}} Q_{n,L})^\natural = e^{-\tilde{V}_{n,L}} Q(X_{\delta_{n+1}}, \Phi, C_{n+1}, \mathbf{w}_{n,L}, g_{n,L})$$

Therefore

$$\begin{aligned} T_\lambda \mathcal{S}(K_n, \lambda)^\natural(X_{\delta_{n+1}}, \Phi) = e^{-\tilde{V}_{n,L}} \big(& Q(X_{\delta_{n+1}}, \Phi, C_{n+1}, \mathbf{w}_{n,L}, g_{n,L}) \\ & + \tilde{Q}_n(X_{\delta_{n+1}}, \Phi, \mathbf{v}_{n+1}, C_{n+1}, g_{n,L}) \big) \end{aligned} \tag{4.20}$$

where $\tilde{Q}_n = \hat{Q}^\natural_{n,L}$ is given after a straightforward but lengthy computation by

$$\tilde{Q}_n(X_{\delta_{n+1}}, \Phi, C_{n+1}, \mathbf{v}_{n+1}, g_{n,L}) = g^2_{n,L} \sum_{j=1}^{3} \tilde{Q}^{(j,j)}(\hat{X}_{\delta_{n+1}}, \Phi; C_{n+1}, v^{(4-j)}_{n+1}) \tag{4.21}$$

where

$$\begin{aligned} \tilde{Q}^{(1,1)}(\hat{X}_{\delta_{n+1}}, \Phi; C_{n+1}, u) &= 2 \int_{\hat{X}_{\delta_{n+1}}} dx dy [\Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x)] u(x-y) \\ \tilde{Q}^{(2,2)}(\hat{X}_{\delta_{n+1}}, \Phi; C_{n+1}, u) &= \int_{\hat{X}_{\delta_{n+1}}} dx dy \Big\{ :[\Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x)]^2:_{C_{n+1}} \\ &\qquad + 4:(\Phi(x)\bar{\Phi}(x))(\Phi(y)\bar{\Phi}(y)):_{C_{n+1}} \Big\} u(x-y) \\ \tilde{Q}^{(3,3)}(\hat{X}_{\delta_{n+1}}, \Phi; C_{n+1}, u) &= 4 \int_{\hat{X}_{\delta_{n+1}}} dx dy :\Phi(x)\bar{\Phi}(x)\Phi(x)\bar{\Phi}(y)\Phi(y)\bar{\Phi}(y):_{C_{n+1}} u(x-y) \end{aligned} \tag{4.22}$$

Define

$$F_{Q_n} = \tilde{Q}(C_{n+1}, \mathbf{v}_{n+1}, g_{n,L}) - Q(C_{n+1}, \mathbf{v}_{n+1}, g_{n,L}) \tag{4.23}$$

evaluated on $X_{\delta_{n+1}}, \Phi$.

Then we have from (4.20) and (4.23)

$$\mathcal{E}_1\left(T_\lambda \mathcal{S}(\lambda, K_n)^\natural, F_n\right) = T_\lambda \mathcal{S}(\lambda, K_n)^\natural - F_{Q_n} e^{-\tilde{V}_{n,L}} = e^{-\tilde{V}_{n,L}} Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n,L}) \quad (4.24)$$

which shows that Q is stable under RG evolution and verifies (4.16). It remains to show that the chosen perturbative relevant part F_{Q_n} given by (4.23) is of the form (3.15) and thus suitable for extraction.

To compute the difference in (4.23) we will make use of the following *localization formulae*

$$\begin{aligned} &\Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x) \\ &\quad = \Phi(x)\bar{\Phi}(x) + \Phi(y)\bar{\Phi}(y) - (\Phi(x) - \Phi(y))(\bar{\Phi}(x) - \bar{\Phi}(y)) \end{aligned} \quad (4.25)$$

$$\begin{aligned} &(\Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x))^2 + 4(\Phi(x)\bar{\Phi}(x))(\Phi(y)\bar{\Phi}(y)) \\ &\quad = 4[(\Phi\bar{\Phi})^2(x) + (\Phi\bar{\Phi})^2(y)] - (\Phi(x) - \Phi(y))(\bar{\Phi}(x) - \bar{\Phi}(y))(\Phi(x) + \Phi(y)) \\ &\qquad \times (\bar{\Phi}(x) + \bar{\Phi}(y)) - 3[(\Phi\bar{\Phi})(x) - (\Phi\bar{\Phi})(y)]^2 \end{aligned} \quad (4.26)$$

that are immediate to check. We get

$$\begin{aligned} F_{Q_n}(X_{\delta_{n+1}}) &= 2g_{n,L}^2 \int_{\hat{X}_{\delta_{n+1}}} dxdy\Big[(\Phi\bar{\Phi})(x) + (\Phi\bar{\Phi})(y)\Big] v^{(3)}(x-y) \\ &\quad + 4g_{n,L}^2 \int_{\hat{X}_{\delta_{n+1}}} dxdy\Big[{:}(\Phi\bar{\Phi})^2{:}_{C_{n+1}}(x) + {:}(\Phi\bar{\Phi})^2{:}_{C_{n+1}}(y)\Big] v^{(2)}(x-y) \end{aligned} \quad (4.27)$$

Note that due to supersymmetry there is no field independent part in F_{Q_n}. We can write $F_{Q_n}(X_{\delta_{n+1}})$ as:

$$F_{Q_n}(X_{\delta_{n+1}}) = \sum_{\Delta_{\delta_{n+1}} \subset X_{\delta_{n+1}}} F_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \quad (4.28)$$

where

$$F_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = 4g_{n,L}^2 F_{Q_n}^{(2)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) + 2g_{n,L}^2 F_{Q_n}^{(1)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \quad (4.29)$$

and

$$F_{Q_n}^{(m)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx{:}(\Phi\bar{\Phi})^m(x){:}_{C_{n+1}} f_{Q_n}^{(m)}(x, X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \quad (4.30)$$

with

$$f_{Q_n}^{(m)}(x, X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \begin{cases} \int_{\Delta_{\delta_{n+1}}} dy v^{(m')}(x-y), & X_{\delta_{n+1}} = \Delta_{\delta_{n+1}} \\ \int_{\Delta'_{\delta_{n+1}}} dy v^{(m')}(x-y), & X_{\delta_{n+1}} = \Delta_{\delta_{n+1}} \cup \Delta'_{\delta_{n+1}}, \text{ connected} \end{cases} \quad (4.31)$$

and $m' = 4 - m$

$$V(F_{Q_n}, \Delta_{\delta_{n+1}}) = \sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = 4g_{n,L}^2 \sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F_{Q_n}^{(2)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}})$$
$$+ 2g_{n,L}^2 \sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F_{Q_n}^{(1)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \tag{4.32}$$

where we have used (3.13), (3.14) and (3.15) for the first equality.

In the following we will use the fact that the $v_{n+1}^{(j)}(x-y)$, $1 \le j \le 3$ vanish for $|x-y| \ge 1$. This follows from the fact that $\Gamma_{n,L}(x-y)$ appears as a factor in the expression (4.16) for $v_{\delta_{n+1}}^{(j)}(x-y)$ and $\Gamma_{n,L}$ has range 1. Returning to (4.32) we have

$$\sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F_{Q_n}^{(m)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx {:}(\Phi\bar{\Phi})^m(x){:}_{C_{n+1}} \Bigg[\int_{\Delta_{\delta_{n+1}}} dy v_{n+1}^{(m')}(x-y)$$
$$+ \sum_{\substack{\Delta'_{\delta_{n+1}} \neq \Delta_{\delta_{n+1}} \\ (\Delta_{\delta_{n+1}}, \Delta'_{\delta_{n+1}}) \text{ connected}}} \int_{\Delta'_{\delta_{n+1}}} dy v_{n+1}^{(m')}(x-y) \Bigg]$$

On the r.h.s. use $v_{n+1}^{(m')}(x-y) = 0$ for $|x-y| \ge 1/2$ to extend the sum on $\Delta'_{\delta_{n+1}}$ to *all* the $\Delta'_{\delta_{n+1}} \neq \Delta_{\delta_{n+1}}$. We then get

$$\sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F_{Q_n}^{(m)}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx {:}(\Phi\bar{\Phi})^m(x){:}_{C_{n+1}} \int dy v^{(m')}(x-y)$$

Hence from (4.32) and above we get

$$V(F_{Q_n}, \Delta_{\delta_{n+1}}) = a_n g_{n,L}^2 \int_{\Delta_{\delta_{n+1}}} dx {:}(\Phi\bar{\Phi})^2(x){:}_{C_{n+1}} + b_n g_{n,L}^2 \int_{\Delta_{\delta_{n+1}}} dx (\Phi\bar{\Phi})(x) \tag{4.33}$$

where

$$a_n = 4 \int_{(\delta_{n+1}\mathbb{Z})^3} dy\ v_{n+1}^{(2)}(y), \qquad b = 2 \int_{(\delta_{n+1}\mathbb{Z})^3} dy\ v_{n+1}^{(3)}(y) \tag{4.34}$$

□

Remark a_n and b_N are well defined since the v_{n+1}^j have compact support. They are positive and their properties are discussed in Lemma 5.12 of Sect. 5.

The Exact RG Map f_{n+1} *for* $K_n = Q_n e^{-V_n} + R_n$

$$K_n \mapsto K_{n+1} = f_{n+1,K}(\lambda, K_n, V_n)|_{\lambda=1} = \mathcal{E}(\mathcal{S}(\lambda, K_n)^{\natural}, F_n(\lambda))|_{\lambda=1} \tag{4.35}$$

induces an evolution of the remainder R_n which is studied by Taylor series around $\lambda = 0$ with remainder written using the Cauchy formula:

$$f_{n+1,K}(\lambda = 1) = \sum_{j=0}^{3} \frac{f_{n+1,K}^{(j)}(0)}{j!} + \frac{1}{2\pi i} \oint_\gamma \frac{d\lambda}{\lambda^4(\lambda - 1)} f_{n+1,K}(\lambda)$$

The terms $j=0,1,2$ are the second order part $f^{(\le 2)}_{n+1,K}$. In the $j=3$ term there are no terms mixing R_n with Q_n, P_n because of the λ^3 in front of R_n. Therefore it splits $\frac{f_K^{(3)}(0)}{3!}=R_{n+1,1}+R_{n+1,2}$ into the third order derivative at $R_n=0$, which we write using the Cauchy formula as

$$R_{n+1,1}\equiv R_{n+1,\text{main}}=\frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4}\mathcal{E}\Big(\mathcal{S}(\lambda,Q_n e^{-V_n})^\natural, F_{Q_n}(\lambda)\Big) \tag{4.36}$$

and terms linear in R_n:

$$\begin{aligned} R_{n+1,2}\equiv R_{n+1,\text{linear}}&=\big(\mathcal{S}_1 R_n\big)^\natural - F_{R_n}e^{-\tilde V_{L,n}}\\ \mathcal{S}_1 R_n(Z_{\delta_{n+1}})&=\sum_{X_{\delta_{n+1}}:L^{-1}\bar X^L_{\delta_{n+1}}=Z_{\delta_{n+1}}} e^{-\tilde V_{n,L}(Z_{\delta_{n+1}}\backslash L^{-1}X_{\delta_{n+1}})}R_{n,L}(L^{-1}X_{\delta_{n+1}}) \end{aligned} \tag{4.37}$$

The remainder term in the Taylor expansion is

$$R_{n+1,3}=\frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4(\lambda-1)}\mathcal{E}(\mathcal{S}(\lambda,K_n)^\natural, F_n(\lambda)) \tag{4.38}$$

In Proposition 4.1 the coupling constant in $Q^{(\le 2)}_{n+1}$ is not the same as the coupling constant in $V^{(\le 2)}_{n+1}$. Furthermore, the coupling constant in $V^{(\le 2)}_{n+1}$ will further change because of the contribution from F_R. To take this into account we introduce

$$\begin{aligned} V_{n+1}(\Delta_{\delta_{n+1}})&=V(\Delta_{\delta_{n+1}},C_{n+1},g_{n+1},\mu_{n+1})\\ g_{n+1}&=L^\varepsilon g_n(1-L^\varepsilon a_n g_n)+\xi_n(u_n)\\ \mu_{n+1}&=L^{\frac{3+\varepsilon}{2}}\mu_n-L^{2\varepsilon}b_n g_n^2+\rho_n(u_n) \end{aligned} \tag{4.39}$$

where $u_n=(g_n,\mu_n,R_n)$ and the remainders $\xi_n(u_n)$, $\rho_n(u_n)$ anticipate the effects of a yet-to-be-specified F_{R_n}. Then we set

$$R_{n+1,4}=e^{-V_{n+1}}Q(C_{n+1},\mathbf{w}_{n+1},g_{n+1})-e^{-\tilde V_{n,L}}Q(C_{n+1},\mathbf{w}_{n+1},g_{n,L}) \tag{4.40}$$

and define

$$\begin{aligned} Q_{n+1}&=Q(C_{n+1},\mathbf{w}_{n+1},g_{n+1})\\ R_{n+1}&=R_{n+1,\text{main}}+R_{n+1,\text{linear}}+R_{n+1,3}+R_{n+1,4}\\ K_{n+1}&=Q_{n+1}e^{-V_{n+1}}+R_{n+1} \end{aligned} \tag{4.41}$$

With these definitions we have obtained the RG map

$$f_{n+1,V}(V_n,K_n)=V_{n+1},\qquad f_{n+1,K}(V_n,K_n)=K_{n+1} \tag{4.42}$$

Definition of F_{R_n}

To complete the RG step we must specify the relevant part F_{R_n} from the remainder R_n. The goal is to choose F_{R_n} so that the map $R_n\to R_{n+1,\text{linear}}$ will be contractive in the following sense. R_n is measured in the norm (2.20), and the kernel norm (2.21), with $\delta=\delta_n$, with a choice of $\mathbf{h}$ and $\mathbf{h}'$ (to be made in Sect. 5). R_{n+1} is measured in the same norms but on the lattice scale δ_{n+1}. We will say that the map is contractive if the size of $R_{n+1,\mathrm{linear}}$ is less than the size of R_n.

F_{R_n} will have the form given in (3.15) with P a supersymmetric polynomial vanishing at $\Phi = 0$. The coefficients α_P will be identified via *normalization conditions* on the small set part of $R_{n+1,\mathrm{linear}}$. This means that certain derivatives with respect to $\Phi = (\varphi, \psi)$ vanish when $\Phi = 0$. That the map in question is contractive when $R_{n+1,\mathrm{linear}}$ is suitably normalized is proven in Sect. 5.

For given coefficients $\tilde{\alpha}_{n,P}(X)$, we define

$$\tilde{F}_{R_n}(X_{\delta_n}, \Phi) = \sum_P \int_{X_{\delta_n}} dx\, \tilde{\alpha}_{n,P}(X_{\delta_n}) P(\Phi(x), \partial_{\delta_n}\Phi(x)) \tag{4.43}$$

$$\tilde{F}_{R_n}(X_{\delta_n}, \Phi) = 0 : \quad X_{\delta_n} \text{is not a small set} \tag{4.44}$$

P runs over the *relevant* monomials which in this model are $P = \Phi\bar{\Phi}$, $(\Phi\bar{\Phi})^2$, $\Phi\partial_{\delta_n,\mu}\bar{\Phi}$, $\partial_{\delta_n,\mu}\Phi\bar{\Phi}$, $\mu \in S$, with the corresponding coefficients $\tilde{\alpha}_P(X_{\delta_n}) = \tilde{\alpha}_{n,2,0}(X_{\delta_n}), \tilde{\alpha}_{n,4}(X_{\delta_n})$, $\tilde{\alpha}_{n,2,\bar{1}}(X_{\delta_n}, \mu), \tilde{\alpha}_{n,2,1}(X_{\delta_n}, \mu)$. The index set S was defined in Sect. 2.1 after (2.1). Note that $P = 1$ is not a relevant monomial in this model: R_n vanishes when $\Phi = 0$ vanishes by hypothesis. Then $R_n^\sharp(X_{\delta_n}, \Phi)$ vanishes when $\Phi = 0$ by supersymmetry, (Lemma 1.1) so that no subtraction is necessary at $\Phi = 0$.

Choose the coefficients $\tilde{\alpha}_{n,P}$ so that

$$J_n = R_n^\sharp - \tilde{F}_{R_n} e^{-\tilde{V}_n} \tag{4.45}$$

is normalized (details are given below). Note that $J(X_{\delta_n}, 0) = 0$. We define the relevant part, supported on small sets, by

$$F_{R_n}(Z_{\delta_{n+1}}, \Phi) = \sum_{\substack{X_{\delta_{n+1}}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}} = Z_{\delta_{n+1}}}} \tilde{F}_{R_n,L}(L^{-1}X_{\delta_{n+1}}, \Phi) = \sum_{\substack{X_{\delta_n}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_n} = Z_{\delta_n}}} \tilde{F}_{R_n}(X_{\delta_n}, S_L\Phi) \tag{4.46}$$

F_{R_n} is supported on small sets by construction. From the definition of $R_{n+1,\mathrm{linear}}$ in (4.37) we get

$$\begin{aligned} R_{n+1,\mathrm{linear}}(Z_{\delta_{n+1}}) = &\sum_{\substack{X_{\delta_{n+1}}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}} = Z_{\delta_{n+1}}}} e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus L^{-1}X_{\delta_{n+1}})} J_{n,L}(L^{-1}X_{\delta_{n+1}}) \\ &+ \sum_{\substack{X_{\delta_{n+1}}:\text{large sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}} = Z_{\delta_{n+1}}}} e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus L^{-1}X_{\delta_{n+1}})} J_{n,L}(L^{-1}X_{\delta_{n+1}}) \end{aligned} \tag{4.47}$$

Therefore the first sum in R_{linear} is also normalized because normalization as defined below is preserved under multiplication by smooth functionals of Φ and rescaling.

Substitution of (4.43) in (4.46) shows that F_{R_n} is of the form required in (3.15). We have

$$F_{R_n}(Z_{\delta_{n+1}}, \Phi) = \sum_P \int_{Z_{\delta_{n+1}}} dx\, \alpha_{n,P}(Z_{\delta_{n+1}}, x) P(\Phi(x), \partial_{\delta_{n+1}}\Phi(x)) \tag{4.48}$$

where

$$\alpha_{n,P}(Z_{\delta_{n+1}},x)=\sum_{\substack{X_{\delta_{n+1}}\ \text{small set}\\ L^{-1}\bar{X}^{L}_{\delta_{n+1}}=Z_{\delta_{n+1}}}}\tilde{\alpha}_{n,P}(X_{\delta_n})L^{-[P]+3}1_{L^{-1}X_{\delta_{n+1}}}(x) \tag{4.49}$$

In (4.49) 1_X is the characteristic function of the set X. Note that $X_{\delta_{n+1}}$ fixes X_{δ_n} by restriction by our construction of polymers in Sect. 1.3. $[P]$ is the dimension of the monomial P, ($2kd_s$ for $(\Phi\bar{\Phi})^k$ and $2d_s+1$ for $\Phi\partial_{\delta_{n+1}}\bar{\Phi}$). $\alpha_{n,P}(Z_{\delta_{n+1}},x)$ is supported on small sets $Z_{\delta_{n+1}}$ and vanishes if $x\notin Z_{\delta_{n+1}}$.

We now compute V_{F_R} following (3.13). Define

$$\alpha_{n,P}:=\sum_{Z_{\delta_{n+1}}\supset x}\alpha_{n,P}(Z_{\delta_{n+1}},x) \tag{4.50}$$

This is independent of x by translation invariance. In fact given an x it belongs uniquely to a block $\Delta_{\delta_{n+1}}$, since our blocks which are restrictions of half open continuum cubes are always disjoint (see Sect. 1.3). The sum over all polymers containing a block $\Delta_{\delta_{n+1}}$ is independent of $\Delta_{\delta_{n+1}}$ by translation invariance.

From (4.49) and (4.50) we get

$$\alpha_{n,P}=L^{-[P]+3}\sum_{X_{\delta_{n+1}}\ \text{small set}:L^{-1}X_{\delta_{n+1}}\supset x}\tilde{\alpha}_{n,P}(X_{\delta_n}) \tag{4.51}$$

$\alpha_{n,P}=0$ for $P=\Phi\partial_{\delta_{n+1}}\bar{\Phi}$ or $\partial_{\delta_{n+1}}\Phi\bar{\Phi}$ by reflection invariance of polymer activities.

Therefore

$$\begin{aligned}\tilde{V}_L(F_{R_n},\Delta_{\delta_{n+1}})&=\int_{\Delta_{\delta_{n+1}}}dx\left\{\alpha_{n,2,0}\Phi\bar{\Phi}+\alpha_{n,4,0}(\Phi\bar{\Phi})^2\right\}\\&=\int_{\Delta_{\delta_{n+1}}}dx\left\{\rho_n(u_n):\Phi\bar{\Phi}:_{C_{n+1}}+\xi_n(u_n):(\Phi\bar{\Phi})^2:_{C_{n+1}}\right\}\end{aligned} \tag{4.52}$$

where $u_n=(g_n,\mu_n,R_n)$ and

$$\rho_n=\alpha_{n,2,0}+2C_{n+1}(0)\alpha_{n,4,0},\qquad \xi_n=\alpha_{4,0} \tag{4.53}$$

which are formulas for the error terms in (4.39).

Normalization Conditions

By an abuse of notation let 1 denote the constant function in $C^2(X_{\delta_n})$ equal to 1. Similarly let 1^{2p} denote the constant function in $C^2(X^{2p}_{\delta_n})$ equal to 1. We will identify the $C^2(X_{\delta_n})$ function $f(x)=x_\mu$ with x_μ. Note that x_μ is defined with respect to an origin which belongs to X_{δ_n}. Similarly we will identify $C^2(X^2_{\delta_n})$ functions $g_2(x_1,x_2)=x_{1,\mu}$, $g_2(x_1,x_2)=x_{2,\mu}$ with $x_{1,\mu},\ x_{2,\mu}$ respectively.

Suppose the polymer activity $J(X_{\delta_n},\Phi)=J(X_{\delta_n},\varphi,\psi)$ is of degree 0, gauge invariant and supersymmetric. We have the following identities:

$$D^{2,0}J(X_{\delta_n},0,0;1^2)=D^{0,2}J(X_{\delta_n},0,0,;1,1) \tag{4.54}$$

$$D^{2,0}J(X_{\delta_n},0,0;x_{1,\mu})=D^{0,2}J(X_{\delta_n},0,0,;x_\mu,1) \tag{4.55}$$

$$D^{2,0}J(X_{\delta_n},0,0;x_{2,\mu}) = D^{0,2}J(X_{\delta_n},0,0,;1,x_\mu) \tag{4.56}$$

$$D^{2,2}J(X_{\delta_n},0,0;1,1,1^2) = 2D^{0,4}J(X_{\delta_n},0,0;1,1,1,1) \tag{4.57}$$

$$D^{4,0}J(X_{\delta_n},0,0;1^4) = 0 \tag{4.58}$$

where the field derivatives are taken according to (2.12). The identities (4.54)–(4.57) follow by expanding $J(X_{\delta_n},\Phi)$ in the fields, retaining a degree 4 supersymmetric polynomial in Φ and $\partial_{\delta_n}\Phi$ which is all that enters into the computation. Then express it in the Grassmann representation (1.85). (4.58) is trivial. Because J is of degree 0 the only term that survives for the computation of (4.58) is of the form $\int_{X^4_{\delta_n}} dx\, a(x_1,x_2,x_3,x_4)\psi(x_1)\bar\psi(x_2)\psi(x_3)\bar\psi(x_4)$ where the kernel a is antisymmetric in x_1,x_3 and in x_2,x_4. The integral vanishes if we replace the Grassmann piece by 1^4. Derivatives on the Grassmann fields annihilate 1^4.

We say that a degree 0, gauge invariant, supersymmetric polymer activity $J(X_{\delta_n},\Phi) = J(X_{\delta_n},\varphi,\psi)$ with $J(X_{\delta_n},0)=0$ is *normalized* if, for all small sets X_{δ_n},

$$\begin{aligned}
D^{2,0}J(X_{\delta_n},0,0;1^2) &= D^{0,2}J(X_{\delta_n},0,0,;1,1) = 0\\
D^{2,0}J(X_{\delta_n},0,0;x_{1,\mu}) &= D^{2,0}J(X_{\delta_n},0,0;x_{2,\mu}) = 0\\
D^{0,2}J(X_{\delta_n},0,0;1,x_\mu) &= D^{0,2}J(X_{\delta_n},0,0;x_\mu,1) = 0\\
2D^{0,4}J(X_{\delta_n},0,0;1,1,1,1) &= D^{2,2}J(X_{\delta_n},0,0;1,1,1^2) = 0
\end{aligned} \tag{4.59}$$

Determining Coefficients from (4.59)

We will apply the normalization conditions to $J=J_n$ defined in (4.45). This will determine the dependence of the error terms $\xi_n,\ \rho_n$ on R_n. Lemma 5.17 will show that these terms are bounded by the kernel norm of R_n.

In doing the following computations note that $J_n(X_{\delta_n},0,0)=0$ as shown earlier. Moreover the odd derivatives $D^{0,j}J_n(X_{\delta_n},0;f^{\times j})$, $j=$odd integer, vanish identically by gauge invariance. It is enough to take derivatives with respect to the bosonic fields φ because of the identities stated above, (4.54) et seq. Taking derivatives of (4.45) and remembering that $\tilde F_{R_n}(X_{\delta_n},0)=0,\ \tilde V_n(X_{\delta_n},0)=0$ we get

$$\begin{aligned}
D^{0,2}J_n(X_{\delta_n},0,0;f,\bar f) &= D^{0,2}R_n^\sharp(X_{\delta_n},0,0;f,\bar f) - D^{0,2}\tilde F_{R_n}(X_{\delta_n},0,0;f,\bar f)\\
D^{0,4}J_n(X_{\delta_n},0,0;f_1,\bar f_1,f_2,\bar f_2) &= D^{0,4}R_n^\sharp(X_{\delta_n},0,0;f_1,\bar f_1,f_2,\bar f_2)\\
&\quad + D^{0,4}\tilde F_{R_n}(X_{\delta_n},0,0;f_1,\bar f_1,f_2,\bar f_2)\\
&\quad + 4D^{0,2}\tilde F_{R_n}(X_{\delta_n},0,0;f,\bar f)D^{0,2}\tilde V_n(X_{\delta_n},0,0;f,\bar f)
\end{aligned} \tag{4.60}$$

where the f are complex valued functions in $C^2(X_{\delta_n})$. A variation of φ along f implies that we vary $\bar\varphi$ along $\bar f$. Note that from (4.43)

$$D^{0,2}F_{R_n}(X_{\delta_n},0,0;1,1) = |X_{\delta_n}|\tilde\alpha_{n,2,0}(X_{\delta_n})$$

$$D^{0,2}F_{R_n}(X_{\delta_n},0,0;1,x_\mu) = |X_{\delta_n}|\tilde\alpha_{n,2,\bar 1}(X_{\delta_n},\mu) + \tilde\alpha_{n,2,0}(X_{\delta_n})\int_{X_{\delta_n}} dx\, x_\mu$$

$$D^{0,2}F_{R_n}(X_{\delta_n},0,0;x_\mu,1) = |X_{\delta_n}|\tilde{\alpha}_{n,2,1}(X_{\delta_n},\mu) + \tilde{\alpha}_{n,2,0}(X_{\delta_n})\int_{X_{\delta_n}} dx\, x_\mu$$

$$D^{0,4}F_{R_n}(X_{\delta_n},0,0;1,1,1,1) = 4|X_{\delta_n}|\tilde{\alpha}_{n,4}(X_{\delta_n})$$

Now imposing successively the conditions (4.49) we get

$$\begin{aligned}
\tilde{\alpha}_{n,2,0}(X_{\delta_n}) &= \frac{1}{|X_{\delta_n}|}D^{0,2}R_n^\sharp(X_{\delta_n},0,0;1,1)\\
\tilde{\alpha}_{n,2,\bar{1}}(X_{\delta_n},\mu) &= \frac{1}{|X_{\delta_n}|}D^{0,2}R_n^\sharp(X_{\delta_n},0,0;1,x_\mu) - \frac{1}{|X_{\delta_n}|}\tilde{\alpha}_{2,0}(X_{\delta_n})\int_{X_{\delta_n}} dx\, x_\mu\\
\tilde{\alpha}_{n,2,1}(X_{\delta_n},\mu) &= \frac{1}{|X_{\delta_n}|}D^{0,2}R_n^\sharp(X_{\delta_n},0,0;,x_\mu,1) - \frac{1}{|X_{\delta_n}|}\tilde{\alpha}_{n,2,0}(X_{\delta_n})\int_{X_{\delta_n}} dx\, x_\mu\\
\tilde{\alpha}_{n,4}(X_{\delta_n}) &= \frac{1}{4}\frac{1}{|X_{\delta_n}|}\Big(D^{0,4}R_n^\sharp(X_{\delta_n},0,0;1,1,1,1)\\
&\quad + D^{0,2}\tilde{V}_n(X_{\delta_n},0,0;1,1)D^{0,2}R_n^\sharp(X_{\delta_n},0,0;1,1)\Big)
\end{aligned} \tag{4.61}$$

We remind the reader that RG transformations preserve the invariance of polymer activities under translations, reflections, and rotations which leave the lattice invariant.

5 Estimates

Let $u_n = (g_n, \mu_n, R_n)$. Then $(\mathbf{w}_n, u_n)$ are the coordinates of the measure density in the polymer representation after n successive applications of the RG map f_j, $1 \le j \le n$, of Sect. 4. The $\mathbf{w}_n$ evolve according to $\mathbf{w}_{n+1} = f_{n+1,\mathbf{w}}(\mathbf{w}_n) = \mathbf{v}_{n+1} + \mathbf{w}_{n,L}$ as given in (4.16). This evolution is independent of u_n and is solved in Lemma 5.9 below. The sequence $\{\mathbf{w}_n, u_n\}$ with $u_{n+1} = f_{n+1}(u_n)$, where the solution for $\mathbf{w}_n$ is incorporated in the map f_n, is the RG trajectory. The index n in R_n also indicates that R_n is supported on polymers in $(\delta_n\mathbb{Z})^3$. Correspondingly the norms for Banach spaces of polymer activities given in Sect. 2 are indexed by the lattice spacing δ_n. In this section we first set up a uniformly bounded domain $\mathcal{D}_n$ for u_n. The rest of this section is then devoted to the proof of Theorem 5.1 below. This theorem controls the remainders (ξ_n, ρ_n) in the flow equations (4.39) together with R_{n+1} in (4.41) when u_n belongs $\mathcal{D}_n$. It also gives bounds on g_{n+1} and μ_{n+1}. Theorem 5.1 will provide essential ingredients for the proof (in Sect. 6) of existence of an initial choice of the mass parameter such that there is a uniformly bounded RG trajectory at all scales labelled by n.

The aforementioned domain will be a ball defined with Banach space norms with the center of the ball fixed i.e. independent of n. To this end we first obtain an approximate discrete flow of the coupling constant g_n from the first equation in (4.39) by ignoring the remainder $\xi_n(g_n, \mu_n, R_n)$. The approximate flow equation has n-dependent coefficients. However we show below (Lemma 5.12), with no assumption about the domain $\mathcal{D}_n$ given below, that the positive coefficients a_n converge geometrically as $n \to \infty$ to a constant $a_{c,*} > 0$. This leads us to set up a reference approximate discrete flow of the coupling constant

$$g_{c,n+1} = L^\varepsilon g_{c,n}(1 - L^\varepsilon a_{c,*}g_{c,n}) \tag{5.1}$$

This may be thought of as an approximate flow in an underlying continuum theory. This approximate flow has a nontrivial fixed point, namely

$$\bar{g} = \frac{L^{\varepsilon} - 1}{L^{2\varepsilon} a_{c,*}} > 0 \tag{5.2}$$

The constant $a_{c,*} = a_{c,*}(L, \varepsilon)$ depends on L, ε in such a way that when $\varepsilon \to 0$ with L fixed $a_{c,*}(L, \varepsilon) \to \bar{a}_{c,*}(L)$ which depends only on L. We will assume L large but fixed for the rest of the paper. We then choose ε sufficiently small depending on L.

We have

$$0 < \bar{g} < C_L \varepsilon \tag{5.3}$$

where C_L is a constant which depends only on L. ε is then a measure of smallness of $\bar{g}$.

In the following $O(1)$ denotes a constant *independent of L, ε and n*. Constants C are *independent of ε and n but may depend on L*. These constants may change from line to line. It will not be necessary to keep track of these changes.

The Domain $\mathcal{D}_n$ We will say that $u_n = (g_n, \mu_n, R_n)$ belongs to the domain $\mathcal{D}_n$ if

$$|g_n - \bar{g}| < \nu\bar{g}, \qquad |\mu_n| < \bar{g}^{2-\delta} \tag{5.4}$$

$$|||R_n|||_n < \bar{g}^{11/4-\eta} \tag{5.5}$$

where the constant ν is held in the range $0 < \nu < \frac{1}{2}$, and

$$|||R_n|||_n = \max\{|R_n|_{\mathbf{h}_*,\mathcal{A},\delta_n},\ \bar{g}^2\|R_n\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_n}\} \tag{5.6}$$

We choose $\kappa = \kappa(L)$ as in Lemma 2.1 and $\rho = \rho(L)$ as in Lemma 5.3 (independent of the domain hypothesis). κ, ρ will be held fixed after L has been chosen sufficiently large. $\delta, \eta = O(1) > 0$ are very small fixed numbers, say 1/64, and $h_B = c\bar{g}^{-1/4}$ with $c = O(1) > 0$ a very small number. Furthermore we take $h_{B*} = \rho^{-1/2} + \kappa^{-1/2}$ and choose $m_0 = 9$. $h_F = h_F(L)$ is an ε independent constant which depends on L and is taken to be sufficiently large. (The dependence of h_F on L enters in the proofs of Lemmas 5.15 and 5.16 below.) We recall that $\mathbf{h} = (h_B, h_F)$, $\mathbf{h}_* = (h_{B*}, h_F)$.

Remark

1. Note that condition (5.5) is equivalent to having both

$$\|R_n\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_n} < \bar{g}^{3/4-\eta} \tag{5.7}$$

$$|R_n|_{\mathbf{h}_*,\mathcal{A},\delta_n} < \bar{g}^{11/4-\eta} \tag{5.8}$$

2. In [13], see (5.1)–(5.3) therein, the domain was specified using ε. In contrast here we specify the domain as in [1] by using $\bar{g}$ instead of ε and moreover we enlarge the domain of g_n slightly for technical reasons.

Recall the definitions of $\rho_n(g_n, \mu_n, R_n)$ and $\xi_n(g_n, \mu_n, R_n)$ from (4.53). We will prove in this section

Theorem 5.1 *Let $u_n = (g_n, \mu_n, R_n) \in \mathcal{D}_n$. Let L be large but fixed followed by ε sufficiently small depending on L. $\bar{g}$ is thus sufficiently small. Let $u_{n+1} = f_{n+1}(u_n)$ where f_{n+1} is the RG map of Sect. 4. Then there exist constants C_L independent of n and ε such that*

$$|\xi_n| \le C_L \bar{g}^{11/4-\eta} \tag{5.9}$$

$$|\rho_n| \le C_L \bar{g}^{11/4-\eta} \tag{5.10}$$

$$|||R_{n+1}|||_{n+1} \le L^{-1/4} \bar{g}^{11/4-\eta} \tag{5.11}$$

$$|g_{n+1} - \bar{g}| < 2\nu \bar{g}^{3/2}, \qquad |\mu_{n+1}| < O(1) L^{\frac{3+\varepsilon}{2}} \bar{g}^{2-\delta} \tag{5.12}$$

Remark The lemmas which follow will serve to prove Theorem 5.1. They are organized as in Sect. 5 of [13]. We remark that Lemmas 5.1–5.4, 5.9, and Lemma 5.12 are independent of the domain hypothesis. *All the other lemmas are under the assumption that (g_n, μ_n, R_n) belong to the domain $\mathcal{D}_n$.* Lemmas 5.21, 5.22, 5.23 and 5.26 are the major parts of the program. $R_{n+1,\mathrm{main}}$ is bounded in Lemma 5.21 and this result determines the qualitative form of the bound on the remainder. $R_{n+1,3}$ and $R_{n+1,4}$ are seen, in Lemmas 5.22, 5.23 to be negligible in comparison. $R_{n+1,\mathrm{linear}}$ is the crux of the program and it is bounded in Lemma 5.26. The remaining lemmas are auxiliary results on the way to these lemmas. These auxiliary lemmas implement some of the following principles: Wick constants $C_n(0)$ are uniformly bounded by constants $C = C_L$. In bounds by $G_\kappa, \mathbf{h}, \mathcal{A}$ norms, a fluctuation field ζ contributes a constant $C = O(1)(\rho(L)\kappa(L))^{-1/2}$ and a field φ contributes a constant $O(1)\bar{g}^{-1/4}$. The contributions of these fields as well as of the Grassmann fields ψ, η are controlled by the structure of the norms defined in Sect. 2 (with above choice of $\mathbf{h}, \mathbf{h}_*$) and later in this section ((5.20) et seq). Integration over the Grassman fluctuation fields η is controlled by the Gramm inequality. In bounds by the $\mathbf{h}_*, \mathcal{A}$ norms, fluctuation fields ζ contribute a constant $C = O(1)(\rho(L)\kappa(L))^{-1/2}$ and fields φ contribute $O(1)$. h_{B*} has been adjusted to take care of the constant C above in the contribution of the fluctuation field.

Lattice Taylor Expansions

In the following we will have occasion to estimate the difference of lattice fields at two different points of a hypercubic lattice $(\delta_n\mathbb{Z})^d$. Let f be a lattice function and x, y be two points in the lattice. We write $y - x = \sum_{j=1}^d \delta_n \varepsilon_j h_j e_j$ where $h_j \in \mathbb{Z}_+$, $\varepsilon_j = \mathrm{sign}(y_j - x_j)$ and the e_j are the unit vectors of the lattice. We will express the difference $f(y) - f(x)$ as a sum of forward and backward lattice derivatives of f along segments of a specified lattice path. As usual a forward derivative in the direction e_j is denoted ∂_{δ_n, e_j} and the backward derivative is denoted $\partial_{\delta_n, -e_j}$. Given $j \in \{1, 2, \ldots, d\}$, $s \in \mathbb{Z}_+$ we define $p_j(x - y, s) \in (\delta_n\mathbb{Z})^d$ by

$$p_j(y - x, s) = \sum_{i=1}^{j-1} (y - x, e_i) e_i + \delta_n \varepsilon_j s e_j \tag{5.13}$$

with the convention that if $j = 1$ the sum is empty. Then it is a simple matter to verify that

$$f(y) - f(x) = \delta_n \sum_{j=1}^{d} \sum_{s_j=0}^{h_j-1} \partial_{\delta_n, \varepsilon_j e_j} f(x + p_j(y - x, s_j)) \tag{5.14}$$

By iterating (5.14) we get the second order lattice Taylor expansion

$$f(y)-f(x)=\sum_{j=1}^{d}((y-x),e_j)\partial_{\delta_n,\varepsilon_j e_j}f(x)$$
$$+\delta_n^2\sum_{j,k=1}^{d}\sum_{s_j=0}^{h_j-1}\sum_{s_k=0}^{h_k-1}\partial_{\delta_n,\varepsilon_j e_j}\partial_{\delta_n,\varepsilon_k e_k}f(x+p_k(p_j(y-x,s_j),s_k)) \tag{5.15}$$

Lemma 5.1 *Let $Z_{\delta_n}, X_{\delta_n}$ be connected 1-polymers in $(\delta_n\mathbb{Z})^3$. Let $Y_{\delta_n}=\emptyset$ or $Y_{\delta_n}=L^{-1}X_{\delta_n}\subset Z_{\delta_n}$ such that $\mathrm{vol}(Z_{\delta_n}\setminus Y_{\delta_n})\geq\frac{1}{2}$. Choose any $\gamma=O(1)>0$ and $\kappa=\kappa(L)>0$ as in Lemma 2.1. Let $\bar g$ be sufficiently small so that $0\leq\bar g\leq\kappa^2$. Let $\varphi:\tilde Z_{\delta_n}^{(5)}\to\mathbb{C}$ where $\tilde Z_{\delta_n}^{(5)}$ is Z_{δ_n} with 5-collar attached (see (1.80), (1.81)). Then there exists an $O(1)$ constant which depends on j such that*

$$\|\varphi\|^j_{C^2(Z_{\delta_n})}\leq O(1)2^{|Z|}\bar g^{-\frac{j}{4}}e^{\gamma\bar g\int_{Z_{\delta_n}\setminus Y_{\delta_n}}dy|\varphi(y)|^4}G_\kappa(Z_{\delta_n},\varphi) \tag{5.16}$$

For $Y_{\delta_n}=\emptyset$ the above bound holds without the factor $2^{|Z|}$.

Proof This is the lattice analogue of Lemma 5.1 in [13]. The proof reposes on the Hölder inequality and the lattice Sobolev inequality (see [8], Appendix B for an elementary proof). Let $x\in Z_{\delta_n}$. Write

$$\varphi(x)=\frac{1}{\mathrm{vol}(Z_{\delta_n}\backslash Y_{\delta_n})}\int_{Z_{\delta_n}\backslash Y_{\delta_n}}dy(\varphi(y)+\varphi(x)-\varphi(y))$$

and bound

$$|\varphi(x)|\leq\frac{1}{\mathrm{vol}(Z_{\delta_n}\backslash Y_{\delta_n})}\int_{Z_{\delta_n}\backslash Y_{\delta_n}}dy|\varphi(y)|+\frac{1}{\mathrm{vol}(Z_{\delta_n}\backslash Y_{\delta_n})}\int_{Z_{\delta_n}\backslash Y_{\delta_n}}dy|\varphi(x)-\varphi(y)|$$

We bound the first term on the right hand side by $O(1)\|\varphi\|_{L^4(Z_{\delta_n}\backslash Y_{\delta_n})}$. To bound the second term we write the difference $\varphi(y)-\varphi(x)$ as a sum of lattice derivatives along the segments of the path as in (5.14). Any connected polymer Z_{δ_n} as defined in Sect. 1.3 can be represented as $Z_{\delta_n}=Z\cap(\delta_n\mathbb{Z})^3$ where Z is a connected continuum polymer. If Z_{δ_n} is a block (unit cube) then the path $p_j(y-x,s_j)$ in (5.14) lies entirely in Z_{δ_n}. If Z_{δ_n} is not a block then it decomposes as a connected union of blocks. If x,y are not in the same block then it suffices to consider the case when they are in adjacent components. We pick a point z_0 in the intersection of the closures, write $f(x)-f(y)=(f(x)-f(z_0))+(f(z_0)-f(y))$ and use the above first order taylor expansion for each summand. The estimates below remain valid. Therefore it is sufficient to consider the case Z_{δ_n} is a block. From

$$\varphi(y)-\varphi(x)=\delta_n\sum_{j=1}^{3}\sum_{s_j=0}^{h_j-1}\partial_{\delta_n,\varepsilon_j e_j}\varphi(x+p_j(y-x,s_j)) \tag{5.17}$$

we get the bound

$$|\varphi(y)-\varphi(x)|\leq\sum_{j=1}^{3}\delta_n|h_j|\sup_{z\in Z_{\delta_n}}|\partial_{\delta_n,\varepsilon_j e_j}\varphi(z)|\leq 3\delta_n(\max_j|h_j|)\max_j\sup_{z\in Z_{\delta_n}}|\partial_{\delta_n,\varepsilon_j e_j}\varphi(z)|$$

$$\leq O(1)|y-x|\,\|\varphi\|_{Z_{\delta_n},1,5} \tag{5.18}$$

where in the last step we have used the lattice Sobolev embedding theorem. We have $|y - x| \leq |Z|$. Putting the above bounds together we get

$$|\varphi(x)| \leq 0(1)|Z|(\|\varphi\|_{L^4(Z_{\delta_n}\setminus Y_{\delta_n})} + \|\varphi\|_{Z_{\delta_n},1,5})$$

We also have for $k = 1, 2$, $|\partial^k_{\delta_n,\mu_1,\dots,\mu_k}\varphi(x)| \leq \|\varphi\|_{Z_{\delta_n},1,5}$ by Sobolev embedding. Therefore combining with the previous inequality we get

$$\|\varphi\|_{C^2(Z_{\delta_n})} \leq 0(1)|Z|(\|\varphi\|_{L^4(Z_{\delta_n}\setminus Y_{\delta_n})} + \|\varphi\|_{Z_{\delta_n},1,5})$$

Hence

$$\begin{aligned}\|\varphi\|^j_{C^2(Z_{\delta_n})} &\leq 0(1)^j|Z|^j(\|\varphi\|^j_{L^4(Z_{\delta_n}\setminus Y_{\delta_n})} + \|\varphi\|^j_{Z_{\delta_n},1,5})\\ &\leq C(j)\,2^{|Z|}\bar{g}^{-j/4}e^{\gamma\bar{g}\int_{Z_{\delta_n}\setminus Y_{\delta_n}} dy\,\varphi^4(y)}G_\kappa(Z_{\delta_n},\varphi)\end{aligned}$$

where $C(j)$ is an O(1) constant that depends on j. We have used the hypothesis that $\bar{g}$ is sufficiently small so that $0 \leq \bar{g} \leq \kappa^2$. This proves the bound (5.16). We now prove the statement following (5.16). Let $Y_{\delta_n} = \emptyset$. For $x \in Z_{\delta_n}$ pick the unit block $\Delta_{\delta_n} \subset Z_{\delta_n}$, $\Delta_{\delta_n} \ni x$. We have

$$|\varphi(x)| \leq \int_{\Delta_{\delta_n}} dy\,|f(y)| + \int_{\Delta_{\delta_n}} dy\,|\varphi(x)-\varphi(y)|$$

Proceeding as before the first term is bounded by the $L^4(\Delta_{\delta_n})$ norm which is less than the $L^4(Z_{\delta_n})$ norm. The second term is bounded as before except that since $x, y \in \Delta_{\delta_n}$ we have $|x-y| \leq O(1)$. The rest is as before. □

In effecting the fluctuation map in Sect. 3.1 we created polymer activities which depended separately on Φ and the fluctuation field ξ. The following lemmas will enable us to estimate the contributions of the bosonic fluctuation field ζ at various steps. Define a large field regulator for the bosonic fluctuation field $\zeta : \tilde{X}^5_{\delta_n} \to \mathbf{C}$

$$\tilde{G}_{\kappa,\rho}(X_{\delta_n},\zeta) = e^{\rho\|\zeta\|^2_{L^2(X_{\delta_n})}}G_\kappa(X_{\delta_n},\zeta), \quad \rho,\kappa > 0 \tag{5.19}$$

κ is chosen as in Lemma 2.1 and is held sufficiently small. The choice of $\rho > 0$ is dictated by Lemma 5.3 below.

Lemma 5.2 *For any $x \in X_{\delta_n}$*

$$|\zeta(x)|^j \leq C_{\rho,j,\kappa}\tilde{G}_{\kappa,\rho}(X_{\delta_n},\zeta) \tag{5.20}$$

where $C_{\rho,j} = (\rho^{-1/2} + \kappa^{-1/2})^j O(1)$ and $O(1)$ depends on j. We have isolated out the ρ, κ dependence in the bound.

Proof The proof follows the lines of the proof of Lemma 5.1 for the case $Y_{\delta_n} = \emptyset$. Take the unit block $\Delta_{\delta_n} \subset X_{\delta_n}$ such that $\Delta_{\delta_n} \ni x$. We replace the L^4 norm by the L^2 norm in the appropriate place and estimate $|\zeta(x) - \zeta(y)|$ with $x, y \in \Delta_{\delta_n}$ as before now using the regulator $\tilde{G}_{\kappa,\rho}$. □

The parameter $\rho > 0$ is chosen such that the following Lemma 5.3 holds. ρ depends on L.

Lemma 5.3 *Let $\kappa > 0$ be chosen as in Lemma 2.1. Then there exists $\rho_0 = \rho_0(L) > 0$ independent of n such that for all ρ, $0 < \rho \le \rho_0$*

$$\int d\mu_{\Gamma_n}(\zeta)\tilde{G}_{\kappa,\rho}(X_{\delta_n},\zeta) \le 2^{|X_{\delta_n}|} \tag{5.21}$$

Lemma 5.3 is proved in the same way as Lemma 2.1.

We introduce a new intermediate large field regulator which combines the ones introduced earlier

$$\hat{G}_{\kappa,\rho}(X_{\delta_n},\zeta,\varphi) = G_\kappa(X_{\delta_n},\zeta+\varphi)G_\kappa(X_{\delta_n},\varphi)\tilde{G}_{\kappa,\rho}(X_{\delta_n},\zeta) \tag{5.22}$$

Lemma 5.4 *Let κ, ρ be chosen as in Lemma 2.1 and Lemma 5.3 respectively. Then we have*

$$\int d\mu_{\Gamma_n}(\zeta)\hat{G}_{\kappa,\rho}(X_{\delta_n},\zeta,\varphi) \le 2^{|X_{\delta_n}|}G_{3\kappa}(X_{\delta_n},\varphi) \tag{5.23}$$

Proof The proof follows from an application of the Hölder inequality and Lemmas 2.1, 5.3. □

Intermediate Norms We will set up some additional norms to help us control intermediate steps where we encounter polymer activities which are functions of the four separate fields $\varphi, \zeta, \psi, \eta$. These norms supplement the basic norms defined in Sect. 2, (2.12)–(2.19).

Let $\tilde{\Omega}(X_{\delta_n})$ be the Grassmann algebra with (bosonic) coefficients in $\mathcal{F}(X_{\delta_n})$ generated by $\psi(x)$, $\bar{\psi}(x)$, $\eta(x)$, $\bar{\eta}(x)$ for all $x \in X_{\delta_n}$. We assign to $\eta, \bar{\eta}$ the same degrees as for $\psi, \bar{\psi}$. This is a graded algebra and $\tilde{\Omega}^0(X_{\delta_n})$ denotes the subalgebra of degree 0 elements. Note that $\Omega^0(X_{\delta_n}) \subset \tilde{\Omega}^0(X_{\delta_n})$. Consider any polymer activity $\tilde{K}(X_{\delta_n},\varphi,\zeta,\psi,\eta) \in \tilde{\Omega}^0(X_{\delta_n})$. Let $I \subset \{1,2,\ldots,p\}$ and $J \subset \{1,2,\ldots,p\}$. Define $I^c = \{1,2,\ldots,p\} \setminus I$. We introduce the abbreviated notation $\mathbf{x}_I := (x_{i_1},\ldots,x_{i_{|I|}})$ where for $i_j \in I$ for $j = 1,\ldots,|I|$. We will refer to the x_{i_j} as the members of $\mathbf{x}_I$. Define $\psi(\mathbf{x}_I) := \psi(x_{i_1}),\ldots,\psi(x_{i_{|I|}})$ and $\frac{\partial}{\partial\psi(\mathbf{x}_I)} := \prod_{j=0}^{|I|-1}\frac{\partial}{\partial\psi(x_{|I|-j})}$. We now define

$$D_F^{2p,IJ}\tilde{K}(X_{\delta_n},\varphi,\zeta,\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c}) := \frac{\partial}{\partial\bar{\eta}(\mathbf{y}_{J^c})}\frac{\partial}{\partial\eta(\mathbf{x}_{I^c})}\frac{\partial}{\partial\bar{\psi}(\mathbf{y}_J)}\frac{\partial}{\partial\psi(\mathbf{x}_I)} \times \tilde{K}(X_{\delta_n},\varphi,\zeta,\psi,\eta)\Big|_{\psi=\eta=0} \tag{5.24}$$

Note that the left hand side is antisymmetric respectively in the members of $\mathbf{x}_I, \mathbf{x}_{I^c}, \mathbf{y}_J, \mathbf{y}_{J^c}$. Let $\mathbf{x} := (\mathbf{x}_I, \mathbf{x}_{I^c})$ and $\mathbf{y} := (\mathbf{y}_J, \mathbf{y}_{J^c})$. Then $\tilde{K}(X_{\delta_n},\varphi,\zeta,\psi,\eta)$ can be represented uniquely as

$$\begin{aligned}\tilde{K}(X_{\delta_n},\varphi,\zeta,\psi,\eta) = \sum_{p\ge 0}\sum_{\substack{I\subset\{1,\ldots,p\}\\ J\subset\{1,\ldots,p\}}}\frac{1}{|I|!\,|I^c|!\,|J|!\,|J^c|!}\int_{X_{\delta_n}^{2p}} d\mathbf{x}d\mathbf{y}\, D_F^{2p,IJ} \\ \times \tilde{K}(X_{\delta_n},\varphi,\zeta,\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c}) \\ \times \psi(\mathbf{x}_I)\bar{\psi}(\mathbf{y}_J)\eta(\mathbf{x}_{I^c})\bar{\eta}(\mathbf{y}_{J^c})\end{aligned} \tag{5.25}$$

Let $g_{IJ} : (\tilde{X}^{(2)}_{\delta_n})^{|I|} \times (\tilde{X}^{(2)}_{\delta_n})^{|J|} \to \mathbf{C}$ such that $g_{IJ}(\mathbf{x}_I, \mathbf{y}_J)$ is antisymmetric respectively in the members of $\mathbf{x}_I$ and those of $\mathbf{y}_J$. Let $h_{I^cJ^c} : (\tilde{X}^{(2)}_{\delta_n})^{|I^c|} \times (\tilde{X}^{(2)}_{\delta_n})^{|J^c|} \to \mathbf{C}$ such that $h_{I^cJ^c}(\mathbf{x}_{I^c}, \mathbf{y}_{J^c})$ is antisymmetric respectively in the members of $\mathbf{x}_{I^c}$ and those of $\mathbf{y}_{J^c}$. The tensor product $g_{IJ} \otimes h_{I^cJ^c}$ maps $(\tilde{X}^{(2)}_{\delta_n})^{|I|} \times (\tilde{X}^{(2)}_{\delta_n})^{|J|} \times (\tilde{X}^{(2)}_{\delta_n})^{|I^c|} \times (\tilde{X}^{(2)}_{\delta_n})^{|J^c|} = (\tilde{X}^{(2)}_{\delta_n})^{2p} \to \mathbf{C}$. By definition $(g_{IJ} \otimes h_{I^cJ^c})(\mathbf{x}_I, \mathbf{x}_{I^c}, \mathbf{y}_J, \mathbf{y}_{J^c}) = g_{IJ}(\mathbf{x}_I, \mathbf{y}_J) h_{I^cJ^c}(\mathbf{x}_{I^c}, \mathbf{y}_{J^c})$. We consider the space of functions g_{IJ} endowed with the $C^2((X_{\delta_n})^{|I|} \times (X_{\delta_n})^{|J|})$ norm. Similarly we consider the space of functions $h_{I^cJ^c}$ endowed with the $C^2((X_{\delta_n})^{|I^c|} \times (X_{\delta_n})^{|J^c|})$ norm. Define

$$\begin{aligned} &D^{2p,IJ,m}\tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0; f^{\times m}, g_{IJ} \otimes h_{I^cJ^c}) \\ &\quad = \int_{X^{2p}_{\delta_n}} d\mathbf{x} d\mathbf{y}\, D^m_B D^{2p,IJ}_F \tilde{K}(X_{\delta_n}, \varphi, \zeta, \mathbf{x}_I, \mathbf{x}_{I^c}, \mathbf{y}_J, \mathbf{y}_{J^c}; f^{\times m}) \\ &\qquad \times (g_{IJ} \otimes h_{I^cJ^c})(\mathbf{x}_I, \mathbf{x}_{I^c}, \mathbf{y}_J, \mathbf{y}_{J^c}) \end{aligned} \tag{5.26}$$

where the derivative D^m_B of the bosonic coefficient is with respect to the field φ (and not the fluctuation field ζ). This defines a multilinear functional on the normed subspace of antisymmetric functions in $C^2((X_{\delta_n})^{|I|} \times (X_{\delta_n})^{|J|}) \times C^2((X_{\delta_n})^{|I^c|} \times (X_{\delta_n})^{|J^c|})$.

The norm of the multilinear functional (5.26) is defined analogously to (2.16), namely

$$\begin{aligned} &\| D^{2p,IJ,m}\tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0)\| \\ &= \sup_{\substack{\|f_j\|_{C^2(X_{\delta_n})} \le 1,\ \forall 1 \le j \le m \\ \|g_{IJ}\|_{C^2(X^{|I|}_{\delta_n} \times (X^{|J|}_{\delta_n})} \le 1 \\ \|h_{I^cJ^c}\|_{C^2(X^{|I^c|}_{\delta_n} \times (X^{|J^c|}_{\delta_n})} \le 1}} \Bigg| \int_{X^{2p}_{\delta_n}} d\mathbf{x} d\mathbf{y} D^m_B D^{2p,IJ}_F \tilde{K}(X_{\delta_n}, \varphi, \zeta, \mathbf{x}_I, \mathbf{x}_{I^c}, \mathbf{y}_J, \mathbf{y}_{J^c}; f^{\times m}) \\ &\qquad \times (g_{IJ} \otimes h_{I^cJ^c})(\mathbf{x}_I, \mathbf{x}_{I^c}, \mathbf{y}_J, \mathbf{y}_{J^c}) \Bigg| \end{aligned} \tag{5.27}$$

In the beginning of this section we specified $\mathbf{h} = (h_F, h_B)$ and $\mathbf{h}_* = (h_F, h_{B*})$.

We define the norms

$$\begin{aligned} \|\tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0)\|_{\mathbf{h}} &= \sum_{p=0}^{\infty} \sum_{m=0}^{m_0} \sum_{\substack{I \subset \{1,\dots,p\} \\ J \subset \{1,\dots,p\}}} \frac{h^m_B}{m!} \frac{h^{2p}_F}{|I|!\,|I^c|!\,|J|!\,|J^c|!} \\ &\quad \times \| D^{2p,IJ,m}\tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0)\| \end{aligned} \tag{5.28}$$

$$\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h},\hat{G}_{\kappa,\rho}} = \sup_{\varphi,\zeta \in \mathcal{F}_{\tilde{X}^{(5)}_{\delta}}} \|\tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0)\|_{\mathbf{h}} \hat{G}^{-1}_{\kappa,\rho}(X_{\delta_n}, \varphi, \zeta) \tag{5.29}$$

$$\begin{aligned} \|\tilde{K}(X_{\delta_n}, 0, \zeta, 0, 0)\|_{\mathbf{h}_*} &= \sum_{p=0}^{\infty} \sum_{m=0}^{m_0} \sum_{\substack{I \subset \{1,\dots,p\} \\ J \subset \{1,\dots,p\}}} \frac{h^m_{B*}}{m!} \frac{h^{2p}_F}{|I|!\,|I^c|!\,|J|!\,|J^c|!} \\ &\quad \times \| D^{2p,IJ,m}\tilde{\tilde{K}}(X_{\delta_n}, 0, \zeta, 0, 0)\| \end{aligned} \tag{5.30}$$

$$\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}_*,\tilde{\mathbf{G}}_{\kappa,\rho}} = \sup_{\zeta \in \mathcal{F}_{\tilde{X}^{(5)}_{\delta}}} \|\tilde{K}(X_{\delta_n}, 0, \zeta, 0, 0)\|_{\mathbf{h}_*} \tilde{G}^{-1}_{\kappa,\rho}(X_{\delta_n}, \zeta) \tag{5.31}$$

$$|\tilde{K}(X_{\delta_n})|_{\mathbf{h}_*} = \sum_{p=0}^{\infty}\sum_{m=0}^{m_0}\sum_{\substack{I\subset\{1,\dots,p\}\\ J\subset\{1,\dots,p\}}} \frac{h_{B*}^m}{m!}\frac{h_F^{2p}}{|I|!\,|I^c|!\,|J|!\,|J^c|!}$$
$$\times \|D^{2p,IJ,m}\tilde{K}(X_{\delta_n},0,0,0,0)\| \tag{5.32}$$

It is straightforward to prove that the above $\mathbf{h}$ and $\mathbf{h}_*$ norms satisfy the multiplicative property of Proposition 2.1.

Special case: Consider the map $\tilde{\Omega}^0(X_{\delta_n}) \to \Omega^0(X_{\delta_n})$ given by

$$\tilde{K}(X_{\delta_n},\varphi,\zeta,\psi,\eta) = K(X_{\delta_n},\varphi+\zeta,\psi+\eta) \tag{5.33}$$

Norms for K were defined earlier in Sect. 2, see (2.12)–(2.18). On the other hand the $\mathbf{h}$ and $\mathbf{h}_*$ norms of $\tilde{K}$ are defined in (5.28), (5.30) above. We have

Lemma 5.4A *Define* $\hat{\mathbf{h}} := (\frac{h_F}{2}, h_B)$ *and* $\hat{\mathbf{h}}_* := (\frac{h_F}{2}, h_{B*})$. *Then we have for the polymer activity defined in* (5.33)

$$\|\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0)\|_{\hat{\mathbf{h}}} \le \|K(X_{\delta_n},\varphi+\zeta,0)\|_{\mathbf{h}} \tag{5.34}$$

$$\|\tilde{K}(X_{\delta_n},0,\zeta,0,0)\|_{\hat{\mathbf{h}}_*} \le \|K(X_{\delta_n},\zeta,0)\|_{\mathbf{h}_*} \tag{5.35}$$

Proof Let $I \subset \{1,\dots,p\}$ and $J \subset \{1,\dots,p\}$. From the definitions (5.24) and (1.86) we have upto a sign factor

$$D_B^m D_F^{2p,IJ}\tilde{K}(X_{\delta_n},\varphi,\zeta,\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c};f^{\times m})$$
$$= (-1)^\sharp D_B^m D_F^{2p} K(X_{\delta_n},\varphi+\zeta,\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c};f^{\times m}) \tag{5.36}$$

Let $(g_{IJ}\otimes h_{I^cJ^c})(\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c})$ be a test function as defined after (5.25). It is antisymmetric respectively in the members of $\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c}$. From (5.26) and (5.36) we get

$$D^{2p,IJ,m}\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0;f^{\times m},g_{2p,IJ})$$
$$= (-1)^\sharp \int_{X_{\delta_n}^{2p}} d\mathbf{x}d\mathbf{y}\, D_B^m D_F^{2p} K(X_{\delta_n},\varphi+\zeta,\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c};f^{\times m})$$
$$\times (g_{IJ}\otimes h_{I^cJ^c})(\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c}) \tag{5.37}$$

Let $(\mathbf{x}_I,\mathbf{x}_{I^c}) = (x_1,\dots,x_p)$ and $(\mathbf{y}_J,\mathbf{y}_{J^c}) = (y_1,\dots,y_p)$. Write $(g_{IJ}\otimes h_{I^cJ^c})(\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c}) = (g_{IJ}\otimes h_{I^cJ^c})(x_1,\dots,x_p,y_1,\dots,y_p)$. Let S_p be the permutation group of $\{1,\dots,p\}$. Define

$$\mathcal{A}(g_{IJ}\otimes h_{I^cJ^c})(x_1,\dots,x_p,y_1,\dots,y_p)$$
$$= \frac{1}{(p!)^2}\sum_{\sigma,\sigma'\in S_p}(g_{IJ}\otimes h_{I^cJ^c})(x_{\sigma(1)},\dots,x_{\sigma(p)},y_{\sigma'(1)},\dots,y_{\sigma'(p)})$$

Now the coefficient function $D_B^m D_F^{2p} K(X_{\delta_n},\varphi+\zeta,\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c};f^{\times m})$ is antisymmetric in $(\mathbf{x}_I,\mathbf{x}_{I^c}) = (x_1,\dots,x_p)$ and in $(\mathbf{y}_J,\mathbf{y}_{J^c}) = (y_1,\dots,y_p)$. Therefore we can replace $g_{IJ}\otimes h_{I^cJ^c}$ in (5.37) by $\mathcal{A}(g_{IJ}\otimes h_{I^cJ^c})$ and hence

$$\left|D^{2p,IJ,m}\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0;f^{\times m},g_{2p,IJ})\right|$$

$$\leq \|D^{2p,m}K(X_{\delta_n},\varphi+\zeta,0\|\prod_{j=1}^{m}\|f_j\|_{C^2(X_{\delta_n})}\|\mathcal{A}(g_{IJ}\otimes h_{I^cJ^c})\|_{C^2(X^{2p}_{\delta_n})}$$

Now $\|\mathcal{A}(g_{IJ}\otimes h_{I^cJ^c})\|_{C^2(X^{2p}_{\delta_n})}\leq \|g_{IJ}\|_{C^2((X_{\delta_n})^{|I|}\times(X_{\delta_n})^{|J|})}\ \|h_{I^cJ^c}\|_{C^2((X_{\delta_n})^{|I^c|}\times(X_{\delta_n})^{|J^c|})}$. Using this in the previous inequality gives

$$\|D^{2p,IJ,m}\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0)\|\leq\|D^{2p,m}K(X_{\delta_n},\varphi+\zeta,0\| \tag{5.38}$$

Therefore

$$\|\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0)\|_{\hat{\mathbf{h}}}\leq\sum_{p=0}^{\infty}\sum_{m=0}^{m_0}\sum_{\substack{I\subset\{1,\dots,p\}\\ J\subset\{1,\dots,p\}}}\frac{h_B^m}{m!}\frac{2^{-2p}h_F^{2p}}{|I|!\,|I^c|!\,|J|!\,|J^c|!}$$

$$\times\|D^{2p,m}K(X_{\delta_n},\varphi,\zeta,0)\| \tag{5.39}$$

Now

$$\sum_{\substack{I\subset\{1,\dots,p\}\\ J\subset\{1,\dots,p\}}}\frac{1}{|I|!\,|I^c|!\,|J|!\,|J^c|!}=\frac{1}{(p!)^2}2^{2p}$$

Substituting this in the previous inequality and using the definition (2.17) gives

$$\|\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0)\|_{\hat{\mathbf{h}}}\leq\|K(X_{\delta_n},\varphi+\zeta,0)\|_{\mathbf{h}}$$

which proves (5.34). The proof of (5.35) is the same. □

Lemma 5.5 *Let g_n,μ_n belong to $\mathcal{D}_n$. Let Y_{δ_n} be a 1-polymer. Then for $V_n(Y_{\delta_n},\Phi,\xi)=V(Y_{\delta_n},\Phi+\xi,C_n,g_n,\mu_n)$ or $V(Y_{\delta_n},\Phi,S_LC_{n+1},g_n,\mu_n)$, we have*

$$\|e^{-V_n(Y_{\delta_n},\Phi,\xi)}\|_{\mathbf{h}}\leq 2^{|Y_{\delta_n}|}e^{-\frac{\bar{g}}{8}\int_{Y_{\delta_n}}dx|\varphi+\zeta|^4(x)} \tag{5.40}$$

$$\|e^{-V_n(Y_{\delta_n},\Phi,\xi)}\|_{\mathbf{h}_*}\leq 2^{|Y_{\delta_n}|} \tag{5.41}$$

for $\varepsilon>0$ sufficiently small depending on L. In the above norms $\mathbf{h}=(h_B,h_F)$ and $\mathbf{h}_*=(h_{B*},h_F)$ are chosen as in the hypothesis for the domain $\mathcal{D}_{\delta_n}$. Thus $h_F=h_F(L)$, $h_B=c\bar{g}^{-\frac{1}{4}}$ with $c=O(1)$ sufficiently small, and $h_{B*}=\rho^{-1/2}+\kappa^{-1/2}$. Note that h_{B*} depends on L via ρ and κ.

Proof It is sufficient to prove this when Y_{δ_n} is a 1-block Δ_{δ_n}. Because otherwise we can write Y_{Δ_n} as a disjoint union of 1-blocks and write the left hand side as a product over 1-block contributions. Then the multiplicative property of the $\mathbf{h}$ norm (Proposition 2.1) gives the lemma.

From the definition of V in (4.1) we get on undoing the Wick ordering (see (1.32)),

$$V_n(\Delta_{\delta_n},\Phi)=V_{n,u}(\Delta_{\delta_n},\varphi)+2g_n\int_{\Delta_{\delta_n}}dx\varphi\bar{\varphi}(x)\psi\bar{\psi}(x)+\tilde{\mu}_n\int_{\Delta_{\delta_n}}dx\psi\bar{\psi}(x) \tag{5.42}$$

where

$$V_{u,n}(\Delta_{\delta_n},\varphi)=\int_{\Delta_{\delta_n}}dx[g_n(\varphi\bar{\varphi}(x))^2+\tilde{\mu}_n\varphi\bar{\varphi}(x)] \tag{5.43}$$

and $\tilde{\mu}_n = \mu_n - 2g_n C_n(0)$.

By the multiplicative property of the $\mathbf{h}$ norm,

$$\|e^{-V(\Delta_{\delta_n},\Phi)}\|_{\mathbf{h}} \le \|e^{-V_{u,n}(\Delta_{\delta_n},\varphi)}\|_{h_B}\|e^{-2g_n\int_{\Delta_{\delta_n}} dx\varphi\bar{\varphi}(x)\psi\bar{\psi}(x)}\|_{\mathbf{h}}\|e^{-\tilde{\mu}_n\int_{\Delta_{\delta_n}} dx\psi\bar{\psi}(x)}\|_{h_F} \tag{5.44}$$

We estimate each of the factors on the right hand side in turn. We observe that by taking ε sufficiently small we can make $\bar{g}$ as small as necessary since $0 < \bar{g} \le C\varepsilon$. Since g_n, μ_n belong to $\mathcal{D}_n$ and $0 < \nu < \frac{1}{2}$, we have $\frac{\bar{g}}{2} \le g_n \le \frac{3}{2}\bar{g}$ and $\mu = O(\bar{g}^{2-\delta})$. Moreover from (5a) of Theorem 1.1 and (1.56) we have the uniform bound $|C_n(0)| \le C_L$. Therefore $|\tilde{\mu}_n| \le C_L\bar{g}$ with a new constant C_L.

For the first factor on the right hand side of (5.44) we have the bound

$$\|e^{-V_{u,n}(\Delta_{\delta_n},\varphi)}\|_{h_B} \le 2^{\frac{1}{2}|\Delta_{\delta_n}|}e^{-\frac{g_n}{2}\int_{\Delta_{\delta_n}} dx(\varphi\bar{\varphi})^2(x)} \tag{5.45}$$

where $|\Delta_{\delta_n}| = 1$. This can be proved on the lines of the proof of Lemma 5.5 of [13] by substituting there $\bar{g}$ for ε and taking account of the previous observations. Thus

$$V_{u,n}(\Delta_{\delta_n},\varphi) - \frac{g_n}{2}\int_{\Delta_n} dx|\varphi|^4 \ge \frac{\bar{g}}{4}\int_{\Delta_n} dx(|\varphi|^4 - C_L|\varphi|^2)$$

Now $C_L|\varphi|^2 \le \frac{1}{2}(|\varphi|^4 + {C_L}^2)$. *Let $\bar{g}$ be sufficiently small so that $\bar{g}{C_L}^2 \le \bar{g}^{\frac{1}{2}}$.* Using these two observations we get from the previous inequality

$$e^{-V_{u,n}(\Delta_{\delta_n},\varphi)} \le (1 + O(\bar{g}^{\frac{1}{2}}))e^{-\frac{g_n}{2}\int_{\Delta_n} dx|\varphi|^4}$$

The rest of the proof which consists of estimating, for $k \ge 1$, $\frac{h_B^k}{k!}\|D^k e^{-V_{u,n}}\|$ goes through as in the proof of Lemma 5.5 of [13] on replacing ε by $\bar{g}$.

Now consider the second factor in the right hand side of (5.44). From the multiplicative property of the $\mathbf{h}$ norm applied to the series expansion of the exponential we get

$$\|e^{-2g_n\int_{\Delta_{\delta_n}} dx\varphi\bar{\varphi}(x)\psi\bar{\psi}(x)}\|_{\mathbf{h}} \le e^{2g_n h_F^2\int_{\Delta} dx\|\varphi\bar{\varphi}(x)\|_{h_B}}$$

Now $g_n \le \frac{3}{2}\bar{g}$ and $\bar{g}^{\frac{1}{2}} = O(1)h_B^{-2}$. Let $t = \frac{|\varphi(x)|}{h_B}$. Then

$$2g_n h_F^2\|\varphi\bar{\varphi}(x)\|_{h_B} \le O(1)\frac{h_F^2}{h_B^2}(t^2 + t + 1) \le O(1)\frac{h_F^2}{h_B^2}(t^4 + 1)$$

which can be proved by two applications of Hölder's inequality. and therefore for the second factor we have the bound

$$\|e^{2g_n\int_{\Delta_{\delta_n}} dx\varphi\bar{\varphi}(x)\psi\bar{\psi}(x)}\|_{\mathbf{h}} \le 2^{O(1)(h_F/h_B)^2|\Delta_{\delta_n}|}e^{O(1)(h_F/h_B)^2\bar{g}\int_{\Delta_{\delta_n|}} dx(\varphi\bar{\varphi}(x))^2} \tag{5.46}$$

Finally for the third factor we have straightforwardly the bound

$$\|e^{\tilde{\mu}_n\int_{\Delta_{\delta_n}} dx\psi\bar{\psi}(x)}\|_{h_F} \le 2^{h_F^2 C_L\bar{g}|\Delta_{\delta_n}|} \tag{5.47}$$

where we have used the bound $|\tilde{\mu}_n| \le C_L\bar{g}$ (see above).

Put together the bounds for the three factors. In the bound (5.45) use $g_n \ge \frac{\bar{g}}{2}$. Let $\bar{g}$ be sufficiently small (thus making h_B sufficiently large) so that $\max(O(1), C_L)(h_F/h_B)^2 \le \frac{1}{16}$

where $O(1)$ is the constant in (5.46) and C_L is the constant encountered above. This ensures that in the bound (5.46) the exponent $O(1)(h_F/h_B)^2 \leq \frac{1}{16}$. It also ensures that in the bound (5.47) the exponent $h_F^2 C_L \bar{g} \leq \frac{1}{16}$. Thus we have obtained for $\bar{g}$ sufficiently small

$$\|e^{-V(\Delta_{\delta_n},\Phi)}\|_{\mathbf{h}} \leq 2^{|\Delta_{\delta_n}|} e^{-\bar{g}/8\int_\Delta dx(\varphi\bar{\varphi})^2(x)}$$

and the first part of the lemma now follows on invoking the argument in the beginning of the proof. The proof of the second part follows the same lines. □

Lemma 5.6 *Let $p_{n,g}(\Delta_{\delta_n},\xi,\Phi)$ and $p_{n,\mu}(\Delta_{d_n},\xi,\Phi)$ be as given in (4.8). Let g_n,μ_n belong to $\mathcal{D}_n$. Let $h_B = c\bar{g}^{-1/4}$ and h_{B*} be as in the definition of $\mathcal{D}_n$. Recall that $\mathbf{h} = (h_B, h_F)$, and $\mathbf{h}_* = (h_{B*}, h_F)$, where $h_F = h_F(L)$. Let $\kappa = \kappa(L) > 0$ and $\rho = \rho(L) > 0$, be as specified in Lemmas 2.1 and 5.3. Then for any $\gamma = O(1) > 0$, $0 \leq s < 1$ we have constants C_L independent of n and ε but depending on L such that*

$$\begin{aligned}\|p_{n,g}(\Delta_{\delta_n},\varphi,\zeta,0,0)\|_{\mathbf{h}} &\leq C_L\bar{g}^{1/4}(1-s)^{-3/4}\tilde{G}_{\kappa,\rho}(\Delta_{\delta_n},\zeta)\\ &\quad\times G_\kappa(\Delta_{\delta_n},\varphi)e^{\bar{g}(1-s)\gamma\int_{\Delta_{\delta_n}} dx\ (\varphi\bar{\varphi})^2(x)} \end{aligned}\tag{5.48}$$

$$\begin{aligned}\|p_{n,\mu}(\Delta_{\delta_n},\varphi,\zeta,0,0)\|_{\mathbf{h}} &\leq C_L\bar{g}^{7/4-\delta}(1-s)^{-1/2}\tilde{G}_{\kappa,\rho}(\Delta_{\delta_n},\zeta)\\ &\quad\times G_\kappa(\Delta_{\delta_n},\varphi)e^{\bar{g}(1-s)\gamma\int_{\Delta_{\delta_n}} dx\ (\varphi\bar{\varphi})^2(x)} \end{aligned}\tag{5.49}$$

$$\|p_{n,g}(\Delta_{\delta_n},0,\zeta,0,0)\|_{\mathbf{h}_*} \leq C_L\,\bar{g}\tilde{G}_{\kappa,\rho}(\Delta_{\delta_n},\zeta)\tag{5.50}$$

$$\|p_{n,\mu}(\Delta_{\delta_n},0,\zeta,0,0)\|_{\mathbf{h}_*} \leq C_L\bar{g}^{2-\delta}\tilde{G}_{\kappa,\rho}(\Delta_{\delta_n},\zeta)\tag{5.51}$$

Proof $p_{n,g}(\Delta_{\delta_n},\xi,\Phi)$ is given in (4.7). We undo the Wick ordering which produces constants $C_n(0)$ uniformly bounded by constant C_L from Corollary 1.1. We can then write it in the form (5.25) by expanding out in the Grassmann fields. Since it is a local polynomial of degree four we get,

$$\begin{aligned}&p_{n,g}(\Delta_{\delta_n},\varphi,\zeta,\psi,\eta)\\ &\quad=\sum_{p=0}^{2}\sum_{\mathbf{a}}\sum_{I\subset\{1,\dots,2p\}}\int_{\Delta_{\delta_n}} dx\,\tilde{p}^{\mathbf{0},\mathbf{a},I}_{n,g,2p}(\Delta_{\delta_n},\varphi,\zeta,x)\prod_{i\in I}\psi_{a_i}(x)\prod_{i\in I^c}\eta_{a_i}(x)\end{aligned}\tag{5.52}$$

where $\mathbf{0}$ means that $l_i = 0\ \forall i$. We have following the definition of the norm in (5.28) with $\hat{\mathbf{h}}$ replaced by $\mathbf{h}$

$$\begin{aligned}&\|p_{n,g}(\Delta_{\delta_n},\varphi,\zeta,0,0)\|_{\mathbf{h}}\\ &\quad\leq\sum_{p=0}^{2}h_F^{2p}\sup_{\|g_{2p}\|\leq 1}\sum_{\mathbf{a}}\sum_{I\subset\{1,\dots,2p\}}\int_{\Delta_{\delta_n}} dx\,\|\tilde{p}^{\mathbf{0},\mathbf{a},I}_{n,g,2p}(\Delta_{\delta_n},\varphi,\zeta,x)\|_{h_B}|g_{2p}(\mathbf{x})|\\ &\quad\leq\sum_{p=0}^{2}h_F^{2p}\sum_{\mathbf{a}}\sum_{\substack{I\subset\{1,\dots,2n\}\\ |I|\ \text{even}}}\int_{\Delta_{\delta_n}} dx\,\|\tilde{p}^{\mathbf{0},\mathbf{a},I}_{n,g,2p}(\Delta_{\delta_n},\varphi,\zeta,x)\|_{h_B}\end{aligned}\tag{5.53}$$

where $g_{2p}(\mathbf{x}) = g_{2p}(x,x,\dots,x)$ and $\|g_{2p}\|$ is the $C^2(\Delta_{\delta_n}^{2p})$ norm of g_{2p}. $\tilde{p}^{\mathbf{0},\mathbf{a},I}_{n,g,2p}(\Delta_{\delta_n},\varphi,\zeta,x)$ is a polynomial in φ,ζ and every term in the h_B norm of $\tilde{p}_{n,g,2p}$ can be estimated as in the proof of Lemma 5.6 of [13]. Each term carries a factor $g_n = O(\bar{g})$. The fluctuation fields ζ are estimated via Lemma 5.2 and the fields φ via Lemma 5.1. For each field φ we lose $\bar{g}^{-\frac{1}{4}}$. In the $p=0$ term the maximum power of φ in $\tilde{p}_{n,g,p}$ is 3, for $p=1$ the maximum power is 2, and for $p=2$ it is 0. The bound (5.48) now follows as in Lemma 5.6, [13]. The bound (5.49) for $p_{n,\mu}(\Delta_{\delta_n}, \xi, \Phi)$ is proved in the same way. We just have to remember that $\mu_n = O(\bar{g}^{2-\delta})$ from the domain hypothesis, and that the maximum power of the field φ in the $\tilde{p}_{n,\mu,p}$ is 1. The remaining parts are proved in the same way. □

Define $p_n(s) = p_n(s, \Delta_{\delta_n}, \Phi, \xi)$ by $p_n(s) = s p_{n,g} + s^2 p_{n,\mu}$. Then $r_{n,1} = r_{n,1}(\Delta_{\delta_n}, \Phi, \xi)$ defined by (4.9) is given by

$$r_{n,1} = \frac{1}{2}\int_0^1 ds(1-s)^2 e^{-p_n(s)-\tilde{V}_n}\big(-p_n'(s)^3 + 6p_n'(s)p_{n,\mu}\big) \tag{5.54}$$

with $p_n'(s) = \frac{d}{ds}p_n(s) = p_{n,g} + 2sp_{n,\mu}$ and $p_n''(s) = 2p_{n,\mu}$.

Lemma 5.7 *Under the conditions of the domain $\mathcal{D}_n$ there exists a constant C_L independent of n and ε but dependent on L such that*

$$\|r_{n,1}(\Delta_{\delta_n})\|_{h,\hat{G}_{\kappa,\rho}} \le C_L \bar{g}^{3/4} \tag{5.55}$$

$$\|r_{n,1}(\Delta_{\delta_n})\|_{h_*,\tilde{G}_{\kappa,\rho}} \le C_L \bar{g}^{3-\delta} \tag{5.56}$$

Proof Follow the proof of the corresponding Lemma 5.7 of [13]. Write $\tilde{V}_n + p_n(s) = V_{n,1}(s) + V_{n,2}(s)$ where

$$V_{n,1}(s) = V(\Delta_{\delta_n}, \Phi + \xi, C_n, sg_n, s^2\mu_n),$$
$$V_{n,2}(s) = V(\Delta_{\delta_n}, \Phi, S_L C_n, (1-s)g_n, (1-s^2)\mu_n)$$

We have

$$\|r_{n,1}(\Delta_{\delta_n}, \varphi, \zeta, 0, 0)\|_{\mathbf{h}} \le \frac{1}{2}\int_0^1 ds(1-s)^2\|e^{-V_{n,1}(s)}\|_{\mathbf{h}}\|e^{-V_{n,2}(s)}\|_{\mathbf{h}}\Big(\|p'(s)\|_{\mathbf{h}}^3 + 6\|p'(s)\|_{\mathbf{h}}\|p_\mu\|_{\mathbf{h}}\Big)$$

g_n, μ_n belong to $\mathcal{D}_n$. Lemmas 5.5 and 5.6 continue to hold with g_n, μ_n replaced by $sg_n, s^2\mu_n$ or $(1-s)g_n, (1-s^2)\mu_n$. We bound $\|e^{-V_{n,1}(s)}\|_{\mathbf{h}} \le 2$ and $\|e^{-V_{n,2}(s)}\|_{\mathbf{h}} \le 2e^{-(1-s)\frac{\bar{g}}{4}\int_{\Delta_{\delta_n}} dx|\varphi|^4(x)}$. We bound the remaining factor (using Lemma 5.6) by $C_L\bar{g}^{\frac{3}{4}}\hat{G}_{\rho,\kappa}(\Delta_{\delta_n}, \varphi, \zeta)e^{(1-s)\bar{g}3\gamma\int_{\Delta_{\delta_n}} dx|\varphi|^4(x)}$. We put the three bounds together and choose $0 < \gamma < \frac{1}{12}$. This gives the bound (5.55). The proof of (5.56) is similar. □

Lemma 5.8 *Under the conditions for the domain $\mathcal{D}_{\delta_n}$ there exists a constant C_L, independent of n and ε but dependent on L such that*

$$\|P_n(\lambda)\|_{\mathbf{h},\hat{G}_{\kappa,\rho},\mathcal{A},\delta_n} \le C_L|\lambda\bar{g}^{1/4}| \quad \text{for } |\lambda\bar{g}^{1/4}| \le 1 \tag{5.57}$$

$$\|P_n(\lambda)\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A},\delta_n} \le C_L|\lambda\bar{g}^{1-\delta/2}| \quad \text{for } |\lambda\bar{g}^{1-\delta/2}| \le 1 \tag{5.58}$$

Proof This follows on applying Lemmas 5.5, 5.6 and 5.7 to $P_n(\lambda)$ defined in (4.9). □

Estimates for $Q_n e^{-V_n}$

We now turn to the estimate of $Q_n e^{-V_n}$. From (4.4)

$$Q_n(X_{\delta_n}, \Phi) = Q(X_{\delta_n}, \Phi; C_n, \mathbf{w}_n, g) = g_n^2 \sum_{j=1}^{3} Q^{(j,j)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(4-j)}) \tag{5.59}$$

where the $Q^{(m,m)}$ are given in (4.6). Under an iteration, see Proposition 4.1, we have

$$w_n^{(p)} \to w_{n+1}^{(p)} = v_{n+1}^{(p)} + w_{n,L}^{p}$$

where $p = 1, 2, 3$ and the $v_n^{(p)}$ are given in Proposition 4.1. Starting with $w_0^{(p)} = 0$ we get by iterating

$$w_n^{(p)} = \sum_{j=0}^{n-1} v_{n-j,L^j}^{(p)} \tag{5.60}$$

For every integer $n \geq 0$ we consider the Banach spaces $\mathcal{W}_{p,\delta_n}$ of functions $f : (\delta_n \mathbb{Z})^3 \mapsto \mathbf{R}$ with norms $\|\cdot\|_{p,n}$, $p = 1, 2, 3$:

$$\|f\|_{p,n} = \sup_{x \in (\delta_n \mathbb{Z})^3} \left((|x| + \delta_n)^{\frac{6p+1}{4}} |f(x)| \right) \tag{5.61}$$

We define the Banach space $\mathcal{W}_n = \mathcal{W}_{1,n} \times \mathcal{W}_{2,n} \times \mathcal{W}_{3,n}$ consisting of vectors $\mathbf{f} = (f^{(1)}, f^{(2)}, f^{(3)})$, $f^{(p)} : (\delta_n \mathbb{Z})^3 \mapsto \mathbf{R}$ with the norm

$$\|\mathbf{f}\|_n = \max_p \|f^{(p)}\|_n \tag{5.62}$$

Let $\mathbf{w}_n = (w_n^{(1)}, w_n^{(2)}, w_n^{(3)})$ as above.

Lemma 5.9

1. *For L sufficiently large and $\varepsilon > 0$ sufficiently small there exists a constant k_L independent of n and ε such that for all $n \geq 1$,*

$$\|\mathbf{w}_n\|_n \leq k_L/2 \tag{5.63}$$

If we start the sequence $\{\mathbf{w}_n\}_{n \geq 0}$ with $\mathbf{w}_0 \neq 0$, with $\|\mathbf{w}_n\|_{\delta_0} \leq k_L/2$, then

$$\|\mathbf{w}_n\|_n \leq k_L, \quad \forall n \geq 0 \tag{5.64}$$

2. *There exists a function $\mathbf{w}_*$ defined on $\bigcup_{n \geq 0} (\delta_n \mathbb{Z})^3 \subset \mathbb{Q}^3$ such that for every integer $l \geq 0$ held fixed, the sequence $\{\mathbf{w}_n\}_{l \leq n}$ converges to $\mathbf{w}_*$ in the norm $\|\cdot\|_l$ as $n \to \infty$. The convergence rate is given by*

$$\|\mathbf{w}_n - \mathbf{w}_*\|_l \leq \tilde{c}_L L^{-qn} \tag{5.65}$$

where $q > 0$ is the constant in Theorem 1.1 and Corollary 1.1. We have $\mathbf{w}_* = \mathbf{v}_{c*} + \mathbf{w}_{*,L}$ in $\mathcal{W}_l$ for every $l \geq 0$.

Proof

1. Let $m = n - j$ with $0 \le j \le n-1$. By definition $v_m^{(p)} = C_{m,L}^p - C_{m+1}^p$ with pointwise multiplication. Since $C_{m,L} = \Gamma_{m,L} + C_{m+1}$, it follows that $v_m^{(p)}$ has $\Gamma_{m,L}$ as factor. From the finite range property of $\Gamma_{m,L}$ it follows that

$$v_m^{(p)}(x) = 0 : \ |x| \ge 1$$

Theorem 1.1, part (5a), and Corollary 1.1 give uniform bounds on the Γ_m and C_m. Therefore there exists a constant $c_{L,p}$ independent of n such that

$$\|v_m^{(p)}\|_{L^\infty((\delta_m\mathbb{Z})^3)} \le c_{L,p}$$

2. By definition

$$\|v_{n-j,L^j}^{(p)}\|_{p,n} = \sup_{x\in(\delta_n\mathbb{Z})^3}\Big(L^{2d_s j}(|x|+\delta_n)^{\frac{6p+1}{4}}|v_{n-j}^{(p)}(L^j x)|\Big)$$
$$= L^{-j(\frac{6p+1}{4}-2d_s)}\sup_{y\in(\delta_{n-j}\mathbb{Z})^3}\Big((|y|+\delta_{n-j})^{\frac{6p+1}{4}}|v_{n-j}^{(p)}(y)|\Big)$$

Because of the finite range property of $v_{n-j}^{(p)}$ of paragraph 1, we can bound $|y| \le 1$ in the weight factor on the right. Because $n - j \ge 1$ we can bound in the weight factor $\delta_{n-j} \le \delta_1 = L^{-1}$. Therefore on using the bound on $v_{n-j}^{(p)}$ of paragraph 1 we get

$$\|v_{n-j,L^j}^{(p)}\|_{p,n} \le L^{-j(\frac{6p+1}{4}-2d_s)}(1+L^{-1})^{\frac{6p+1}{4}}c_{L,p}$$

We bound the first geometric factor by taking $p = 1$ and $\varepsilon > 0$ very small in $d_s = (3-\varepsilon)/4$. This gives $L^{-j/5}$. This gives the bound

$$\|v_{n-j,L^j}^{(p)}\|_{p,n} \le L^{-j/5}\, c_{L,p}$$

with a new constant $c_{L,p}$ independent of n. Using the above bound we get from (5.60) the bound

$$\|w_n^{(p)}\|_{p,n} \le c_{L,p}\sum_{j=0}^{\infty} L^{-j/5} \le 2c_{L,p}$$

for L sufficiently large. Therefore setting $k_L = 4\max_p c_{L,p}$ we get

$$\|\mathbf{w}_n\|_n \le k_L/2$$

which proves (5.63). Equation (5.64) is a trivial consequence of the above. This proves the first part of the lemma.

3. Let $v_{c*}^{(p)} = C_{c*,L}^p - C_{c*}^p$, with pointwise multiplication, where C_{c*} is the smooth continuum covariance in $\mathbb{R}^3$ of Corollary 1.1. By factoring out $\Gamma_{c*,l}$, Theorem 1.1 and Corollary 1.1 we have that $v_{c*}^{(p)}$ exists in $L^\infty(\mathbb{R}^3)$ and has finite range: $v_{c*}^{(p)}(x) = 0 : |x| \ge 1$. Moreover by Theorem 1.1 and Corollary 1.1 we have, see the proof of Lemma 5.12 for the detailed argument,

$$\|v_m^{(p)} - v_{c*}^{(p)}\|_{L^\infty((\delta_m\mathbb{Z})^3)} \le c_{L,p}L^{-q(m-1)}$$

Define

$$w_*^{(p)} = \sum_{j=0}^{n-1} v_{c*,L^j}^{(p)}$$

Fix any integer $l \geq 0$. Then for $n \geq l$ the (p,l) norm is dominated by the (p,n) norm. Then proceeding as in the first part and using the previous inequality we get

$$\begin{aligned}\|w_n^p - w_*^p\|_{p,l} &\leq \sum_{j=0}^{n-1} \|v_{n-j,L^j}^p - v_{c*,L^j}^p\|_{p,n} \\ &\leq c_{L,p} \sum_{j=0}^{n-1} L^{-j(\frac{6p+1}{4}-2d_s)} \|v_{n-j}^p - v_{c*}^p\|_{L^\infty((\delta_{n-j}\mathbb{Z})^3)} \\ &\leq c_{L,p} \sum_{j=0}^{n-1} L^{-j/5} L^{-q(n-j-1)} \leq c'_{L,p} L^{-qn}\end{aligned}$$

Now take the maximum over p. This proves (5.61) and at the same time the convergence statement of part 2 of the lemma. The last statement of part 2 is trivial to prove. This completes the proof of the second part of the lemma. □

Lemma 5.10 *Under the conditions of the domain $\mathcal{D}_n$ there exists constants $C_{p,L}$ independent of n such that*

$$\|Q_n e^{-V_n}\|_{\mathbf{h},G_\kappa,\mathcal{A}_p,\delta_n} \leq C_{p,L}\, \bar{g}^{1/2} \tag{5.66}$$

$$|Q_n e^{-V_n}|_{\mathbf{h}_*,\mathcal{A}_p,\delta_n} \leq C_{p,L}\, \bar{g}^2 \tag{5.67}$$

Proof

$$Q_n(X_{\delta_n}) e^{-V_n(X_{\delta_n})} = g_n^2 \sum_{m=1}^{3} Q^{(m,m)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(4-m)}) e^{-V_n(X_{\delta_n})} \tag{5.68}$$

$$\left\| Q_n(X_{\delta_n}, \varphi) e^{-V_n(X_{\delta_n},\varphi)} \right\|_{\mathbf{h}} \leq g_n^2 \sum_{m=1}^{3} \left\| Q^{(m,m)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(4-m)}) \right\|_{\mathbf{h}} \left\| e^{-V_n(X_{\delta_n})} \right\|_{\mathbf{h}} \tag{5.69}$$

Here X_{δ_n} is a small set because of the support properties of Q_n. The last factor will be estimated by Lemma 5.5. From (4.6) we have

$$\begin{aligned}&Q^{(3,3)}(\hat{X}_{\delta_n}, \Phi; w_n^{(1)}) \\ &\quad = 4 \int_{\hat{X}_{\delta_n}} dxdy\, :\Phi(x)\bar{\Phi}(x)\Phi(x)\bar{\Phi}(y)\Phi(y)\bar{\Phi}(y):_{C_n} w_n^{(1)}(x-y)\end{aligned} \tag{5.70}$$

We exhibit (5.70) as an element of the Grassmann algebra:

$$\begin{aligned}Q^{(3,3)}(\hat{X}_{\delta_n}, \Phi; w_n^{(1)}) &= Q_0^{(3,3)}(\hat{X}_{\delta_n}, \varphi; w_n^{(1)}) \\ &\quad + \int_{\hat{X}_{\delta_n}} dxdy\, Q_1^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x, y; w_n^{(1)}) :\psi(x)\bar{\psi}(y):_{C_n}\end{aligned}$$

$$
\begin{aligned}
&+\int_{X_{\delta_n}} dx\ Q_2^{(3,3)}(\hat{X}_{\delta_n},\varphi,x;w_n^{(1)}){:}\psi(x)\bar{\psi}(x){:}_{C_n}\\
&+\int_{\hat{X}_{\delta_n}} dxdy\ Q_3^{(3,3)}(\hat{X}_{\delta_n},\varphi,x,y;w_n^{(1)}){:}\psi(x)\bar{\psi}(x)\\
&\times\psi(y)\bar{\psi}(vy){:}_{C_n}
\end{aligned}
\tag{5.71}
$$

where

$$
\begin{aligned}
Q_0^{(3,3)}(\hat{X}_{\delta_n},\varphi;w_n^{(1)}) &= 4\int_{\hat{X}_{\delta_n}} dxdy\ {:}\varphi(x)\bar{\varphi}(x)\varphi(x)\bar{\varphi}(y)\\
&\quad\times\varphi(y)\bar{\varphi}(y){:}_{C_n} w_n^{(1)}(x-y)\\
Q_1^{(3,3)}(\hat{X}_{\delta_n},\varphi,x,y;w_n^{(1)}) &= 4{:}\varphi(x)\bar{\varphi}(x)\varphi(y)\bar{\varphi}(y){:}_{C_n} w_n^{(1)}(x-y)\\
Q_2^{(3,3)}(\hat{X}_{\delta_n},\varphi,x;w_n^{(1)}) &= 4\int_{X'_{\delta_n}} dy{:}(\varphi(x)\bar{\varphi}(y)+\varphi(y)\bar{\varphi}(x))\\
&\quad\times\varphi(y)\bar{\varphi}(y){:}_{C_n} w_n^{(1)}(x-y)\\
Q_3^{(3,3)}(\hat{X}_{\delta_n},\varphi,x,y;w_n^{(1)}) &= 4{:}\varphi(x)\bar{\varphi}(y){:}_{C_n} w_n^{(1)}(x-y)
\end{aligned}
\tag{5.72}
$$

where, denoting with $\Delta(x)$ the block Δ such that $x\in\Delta$, we have

$$
X'_{\delta_n}=\begin{cases}\Delta & \text{if } X_{\delta_n}=\Delta\\ X_{\delta_n}\setminus\Delta(x) & \text{if } X_{\delta_n}=\Delta_1\cup\Delta_2\end{cases}
$$

Undo the Wick ordering, which produces lower order terms with coefficients which are uniformly bounded independent of n by Corollary 1.1. It is therefore enough to estimate with the Wick ordering taken off. We get

$$
\begin{aligned}
&\|Q^{(3,3)}(\hat{X}_{\delta_n},\Phi;w_n^{(1)})\|_{\mathbf{h}}\\
&\quad\le\|Q_0^{(3,3)}(\hat{X}_{\delta_n},\varphi;w_n^{(1)})\|_{h_B}\\
&\qquad+h_F^2\sup_{\|g_2\|_{C^2(\hat{X}^2_{\delta_n})}\le 1}\int_{\hat{X}_{\delta_n}} dxdy\ \|Q_1^{(3,3)}(\hat{X}_{\delta_n},\varphi,x,y;w_n^{(1)})\|_{h_B}|g_2(x,y)|\\
&\qquad+h_F^2\sup_{\|g_2\|_{C^2(\hat{X}^2_{\delta_n})}\le 1}\int_{X_{\delta_n}} dx\ \|Q_2^{(3,3)}(\hat{X}_{\delta_n},\varphi,x;w_n^{(1)})\|_{h_B}|g_2(x,x)|\\
&\qquad+h_F^4\sup_{\|g_4\|_{C^2(\hat{X}^4_{\delta_n})}\le 1}\int_{\hat{X}_{\delta_n}} dxdy\ \|Q_3^{(3,3)}(\hat{X}_{\delta_n},\varphi,x,y;w_n^{(1)})\|_{h_B}|g_4(x,x,y,y)|
\end{aligned}
$$

To estimate the h_B norm of the $Q_j^{(3,3)}$ we apply to (5.72) $h_B^k D^k$, with D the bosonic field derivative, $h_B=c\bar{g}^{-1/4}$ and use Lemma 5.1. Contributions for $k>4$ vanish. We use $g_n=O(\bar{g})$ from the domain hypothesis of Theorem 5.1. We estimate the kernel $w_n^{(1)}(x-y)$ using Lemma 5.9. As a result we get

$$\|Q^{(3,3)}(\hat{X}_{\delta_n},\Phi;w_n^{(1)})\|_{\mathbf{h}} \le C_L\bar{g}^{-3/2}\int_{\hat{X}_{\delta_n}} dxdy\,\frac{1}{(|x-y|+\delta_n)^{7/4}}e^{\bar{g}/4\int_{X_{\delta_n}}dx|\varphi\bar{\varphi}(x)|^2}G_\kappa(X_{\delta_n},\varphi) \tag{5.73}$$

The integral over $\hat{X}_{\delta_n}$ exists and is of $O(1)$ since X_{δ_n} is a small set in $(\delta_n\mathbb{Z})^3$. Therefore

$$\|Q^{(3,3)}(\hat{X}_{\delta_n},\Phi;w_n^{(1)})\|_{\mathbf{h}} \le C_L\bar{g}^{-3/2}e^{\bar{g}/4\int_{X_{\delta_n}}dx|\varphi\bar{\varphi}(x)|^2}G_\kappa(X_{\delta_n},\varphi) \tag{5.74}$$

Next turn to $Q^{(m,m)}$, $m=1,2$. From (4.6)

$$\begin{aligned}
&Q^{(2,2)}(\hat{X}_{\delta_n},\Phi;C,w_n^{(2)})\\
&\quad=-\int_{\hat{X}_{\delta_n}} dxdy[:(\Phi(x)-\Phi(y))(\bar{\Phi}(x)-\bar{\Phi}(y))(\Phi(x)+\Phi(y))(\bar{\Phi}(x)+\bar{\Phi}(y)):_{C_n}\\
&\qquad+:[(\Phi\bar{\Phi})(x)-(\Phi\bar{\Phi})(y)]^2:_{C_n}]w_n^{(2)}(x-y)
\end{aligned}$$

We exhibit this as an element of the Grassmann algebra. This gives

$$\begin{aligned}
&Q^{(2,2)}(\hat{X}_{\delta_n},\Phi;C_n,w_n^{(2)})\\
&\quad=-Q_0^{(2,2)}(\hat{X}_{\delta_n},\varphi;C_n,w_n^{(2)})\\
&\qquad-\int_{\hat{X}_{\delta_n}} dxdy\big[Q_1^{(2,2)}(\hat{X}_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}):(\psi(x)+\psi(y))(\bar{\psi}(x)+\bar{\psi}(y)):_{C_n}\\
&\qquad+Q_2^{(2,2)}(\hat{X}_{\delta_n},\varphi,x,y;C_n,w_n^{(2)},):(\psi(x)-\psi(y))(\bar{\psi}(x)-\bar{\psi}(y)):_{C_n}\\
&\qquad+Q_3^{(2,2)}(\hat{X}_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}):(\psi\bar{\psi}(x)-\psi\bar{\psi}(y)):_{C_n}\\
&\qquad+Q_4^{(2,2)}(\hat{X}_{\delta_n},x,y,w_n^{(2)})\{(:\psi(x)-\psi(y))(\bar{\psi}(x)-\bar{\psi}(y))(\psi(x)+\psi(y))\\
&\qquad\times(\bar{\psi}(x)+\bar{\psi}(y)):_{C_n}\\
&\qquad+:(\psi\bar{\psi}(x)-\psi\bar{\psi}(y))^2):_{C_n})\}\big]
\end{aligned} \tag{5.75}$$

where

$$\begin{aligned}
Q_0^{(2,2)}(\hat{X}_{\delta_n},\varphi;C_n,w_n^{(2)}) &= \int_{\hat{X}_{\delta_n}} dxdy\,w_n^{(2)}(x-y)\\
&\quad\times(:|\varphi(x)-\varphi(y)|^2|\varphi(x)+\varphi(y)|^2:_{C_n}\\
&\quad+:(|\varphi|^2(x)-|\varphi|^2(y))^2:_{C_n})\\
Q_1^{(2,2)}(\hat{X}_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}) &= w_n^{(2)}(x-y):|\varphi(x)-\varphi(y)|^2:_{C_n}\\
Q_2^{(2,2)}(\hat{X}_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}) &= w_n^{(2)}(x-y):|\varphi(x)+\varphi(y)|^2:_{C_n}\\
Q_3^{(2,2)}(\hat{X}_{\delta_n},\varphi,x,y;C_n,w_n^{(2)}) &= 2w_n^{(2)}(x-y):(|\varphi|^2(x)-|\varphi|^2(y)):_{C_n}\\
Q_4^{(2,2)}(\hat{X}_{\delta_n},x,y,w_n^{(2)}) &= w_n^{(2)}(x-y)
\end{aligned} \tag{5.76}$$

The $\|\cdot\|_{\mathbf{h},G_\kappa,\mathcal{A}_p}$ norm estimate for $Q^{(2,2)}(\hat{X}_{\delta_n},\Phi;C,w_n^{(2)})$ reposes on the following principles:

1. Undoing the Wick ordering produces lower order terms with Wick coefficients which together with their derivatives are uniformly bounded independent of n by Corollary 1.1. Moreover by the domain hypothesis $g_n = O(\bar{g})$.

2. By Lemma 5.9, the kernel $w_n^{(2)}$ has the bound $|w_n^{(2)}(x-y)| \leq k_L(|x-y|+\delta_n)^{-13/4}$ where the constant k_L is independent of n.

3. The fields $\varphi(x)$ are estimated by Lemma 5.1. Differences of fields $|\varphi(x)-\varphi(y)|$ are estimated by (5.18). This produces a factor $|x-y|$ which we retain, and majorise the Sobolev factor by the large field regulator. Differences of fields $\varphi\bar{\varphi}(x)-\varphi\bar{\varphi}(y)$ can also be expressed as in (5.17), substituting $\varphi\bar{\varphi}$ for φ. This requires estimating $(\partial_{\delta_n,e_j}\varphi\bar{\varphi})(x+\cdot\cdot)$. We apply the lattice Leibniz which modifies the continuum rule by producing an extra term $\delta_n|\partial_{\delta_n,e_j}\varphi(x+\cdot\cdot)|^2$ (see (5.2), p. 432 of [8]). We estimate the φ by Lemma 5.1, with $\kappa/2$ in the large field regulator. We estimate the gradient pieces by the Sobolev inequality as in (5.18), and then by the large field regulator with $\kappa/2$. We have also produced a factor $|x-y|$ as in (5.18).

Invoking the above principles we get the following bounds for the bosonic coefficients:

$$\begin{aligned}
&\frac{h_B^k}{k!}\|D^k Q_0^{(2,2)}(\hat{X}_{\delta_n},\varphi;C_n,w_n^{(2)})\| \\
&\quad \leq c_L\bar{g}^{-1/2}\int_{\hat{X}_{\delta_n}} dxdy\,(|x-y|+\delta_n)^{-(\frac{13}{4}+2)}e^{\bar{g}/4\int_{X_{\delta_n}}dx|\varphi\bar{\varphi}(x)|^2}G_\kappa(X_{\delta_n},\varphi)
\end{aligned} \tag{5.77}$$

and for $j=1,2,3$

$$\begin{aligned}
&\frac{h_B^k}{k!}\|D^k Q_j^{(2,2)}(\hat{X}_{\delta_n},\varphi;x,y,C_n,w_n^{(2)})\| \\
&\quad \leq c_L\bar{g}^{-1/2}\,(|x-y|+\delta_n)^{-13/4}f_j(x-y)e^{\bar{g}/4\int_{X_{\delta_n}}dx|\varphi\bar{\varphi}(x)|^2}G_\kappa(X_{\delta_n},\varphi)
\end{aligned} \tag{5.78}$$

where the maximum value of k which gives a nonvanising contribution is 4 and

$$\begin{aligned}
f_1(x-y) &= |x-y|^2 \\
f_2(x-y) &= 1 \\
f_3(x-y) &= |x-y| \\
f_4(x-y) &= 1
\end{aligned} \tag{5.79}$$

4. We must estimate the contribution of the fermionic pieces to the $\mathbf{h}$ norm. To this end denote by $F_j(\psi)$ the fermionic factor multiplying $Q_j^{(2,2)}$ in (5.75). Express the differences $\psi(x)-\psi(y)$ by the fermionic analogue of (5.14). We do the same also for $\psi\bar{\psi}(x)-\psi\bar{\psi}(y)$ and then apply the lattice Leibniz rule to $\partial_{\delta_n,\varepsilon_j}\psi\bar{\psi}(x+\cdot)$. We replace the fermionic pieces by the functions g_{2p} on $\hat{X}_{\delta_n}\cup\partial_2\hat{X}_{\delta_n}$ and their lattice derivatives. Corresponding to $F_j(\psi)$ we get the contribution G_j which is a linear form on g_{2p_j}, where $p_j=1$ for $j=1,2,3$ and $p_4=2$. Let $\delta_n h_j$, $h_j\in\mathbf{Z}$ be the component of $y-x$ along the unit vector e_j. We have

$$G_1 = g_2(x,x)+g_2(x,y)+g_2(y,x)+g_2(y,y)$$

$$G_2 = \delta_n^2\sum_{i_1,i_2=1}^{3}\sum_{0\leq s_{i_l}\leq h_{i_l}-1,\,l=1,2}\partial^{(1)}_{\delta_n,e_{i_1}}\partial^{(2)}_{\delta_n,e_{i_2}}g_2(x+p_{i_1}(y-x,s_{i_1}),x+p_{i_2}(y-x,s_{i_2}))$$

$$G_3 = \delta_n \sum_{i=1}^{3} \sum_{s_i=0}^{h_i-1} \Big[\partial^{(1)}_{\delta_n,e_i} g_2(x+p_i(y-x,s_i), x+p_i(y-x,s_i))$$

$$+ \partial^{(2)}_{\delta_n,e_i} g_2(x+p_i(y-x,s_i), x+p_i(y-x,s_i))$$

$$+ \delta_n \partial^{(1)}_{\delta_n,e_i} \partial^{(2)}_{\delta_n,e_i} g_2(x+p_i(y-x,s_i), x+p_i(y-x,s_i)) \Big]$$

$$G_4 = \delta_n^2 \sum_{i_1,i_2=1}^{3} \sum_{0\le s_{i_l}\le h_{i_l}-1,\, l=1,2} \partial^{(1)}_{\delta_n,e_{i_1}} \partial^{(2)}_{\delta_n,e_{i_2}} \Big(g_4(x+p_{i_1}(y-x,s_{i_l}), x+p_{i_2}(y-x,s_{i_2}), x, x)$$

$$+ g_4(x+\cdot\cdot, x+\cdot\cdot, x, y) + g_4(x+p_{i_1}(y-x,s_{i_1}), x+p_{i_2}(y-x,s_{i_2}), y, x)$$

$$+ g_4(x+p_{i_1}(y-x,s_{i_1}), x+p_{i_2}(y-x,s_{i_2}), y, y) \Big) + \cdot\cdot$$

where the superscript on the lattice derivative denotes the argument on which it acts. The omitted terms $\cdot\cdot$ in G_4 comes from the square of the (first order) lattice Taylor expansion of $\psi\bar\psi(x) - \psi\bar\psi(y)$ and then replacing the product of 4 Grassmann fields by the test function g_4. For $j = 1, 2, 3$ we have the bounds

$$|G_j| \le O(1)\tilde f_j(x-y)\|g_2\|_{C^2(\hat X^2_{\delta_n})} \tag{5.80}$$

and for $j = 4$ we have

$$|G_4| \le O(1)\tilde f_4(x-y)\|g_4\|_{C^2(\hat X^4_{\delta_n})} \tag{5.81}$$

where

$$\tilde f_1 = 1, \qquad \tilde f_2 = |x-y|^2, \qquad \tilde f_3 = |x-y|, \qquad \tilde f_4 = |x-y|^2 \tag{5.82}$$

On using the bounds (5.77)–(5.82) we get for $0 \le k \le 4$ and $0 \le p \le 2$

$$\begin{aligned} &h_F^{2p} \frac{h_B^k}{k!} |D^{2p,n} Q^{(2,2)}(\hat X_{\delta_n}, \varphi, 0; f^{\times k}, g_{2p})| \\ &\quad \le c_L \bar g^{-1/2} \int_{\hat X_{\delta_n}} dxdy\, (|x-y|+\delta_n)^{-(\frac{13}{4}-2)} \\ &\qquad \times e^{\bar g/4 \int_{\hat X_{\delta_n}} dx|\varphi\bar\varphi(x)|^2} G_\kappa(X_{\delta_n},\varphi)\|f\|^{\times k}_{C^2(\hat X_{\delta_n})}\|g_{2p}\|_{C^2(\hat X^{2p}_{\delta_n})} \end{aligned} \tag{5.83}$$

For $k > 4$ or $p > 2$ we have vanishing contribution. The integral over $\hat X_{\delta_n}$ exists and gives a contribution of $O(1)$ since X_{δ_n} is a small set in $(\delta_n\mathbb{Z})^3$. Therefore we obtain from the previous inequality

$$\|Q^{(2,2)}(\hat X_{\delta_n}, \varphi, 0)\|_{\mathbf{h}} \le c_L \bar g^{-1/2} e^{\bar g/4 \int_{\hat X_{\delta_n}} dx|\varphi\bar\varphi(x)|^2} G_\kappa(X_{\delta_n},\varphi) \tag{5.84}$$

We can estimate in the same way the case $m = 1$. We have

$$\begin{aligned} &\|Q^{(1,1)}(\hat X_{\delta_n}, \Phi; C, w^{(3)})\|_{\mathbf{h}} \\ &\quad \le c_L \bar g^{-1/2} \int_{\hat X_{\delta_n}} dxdy\, (|x-y|+\delta_n)^{-(\frac{19}{4}-2)} e^{\bar g/4 \int_{X_{\delta_n}} dx|\varphi\bar\varphi(x)|^2} G_\kappa(X_{\delta_n},\varphi) \end{aligned} \tag{5.85}$$

The integral over $\hat{X}_{\delta_n}$ exists and gives a contribution of $O(1)$. Therefore

$$\|Q^{(1,1)}(\hat{X}_{\delta_n}, \Phi; C, w^{(3)})\|_{\mathbf{h}} \leq c_L \bar{g}^{-1/2} e^{\bar{g}/4 \int_{X_{\delta_n}} dx |\varphi\bar{\varphi}(x)|^2} G_\kappa(X_{\delta_n}, \varphi) \tag{5.86}$$

Therefore from (5.69), Lemma 5.5 and the bounds (5.74), (5.84), (5.86) we get

$$\left\| Q_n(X_{\delta_n}) e^{-V_n(X_{\delta_n})} \right\|_{\mathbf{h},\mathbf{G}_\kappa} \leq c_L \bar{g}^{1/2}$$

and since Q_n is supported on small sets we get

$$\|Q_n e^{-V_n}\|_{\mathbf{h},G_\kappa,\mathcal{A}_p} \leq C_{L,p} \bar{g}^{1/2} \tag{5.87}$$

which is (5.66). To prove (5.67) we estimate the r.h.s of (5.69) at $\Phi = 0$ after undoing the Wick ordering, set $\mathbf{h} = \mathbf{h}_*$, and use Lemma 5.5. □

In the following lemma we consider $Q_n(\Phi + \xi) e^{-V_n(\Phi+\xi)}$ as a function of $\varphi, \zeta, \psi, \eta$.

Lemma 5.11 *Under the conditions of the domain $\mathcal{D}_n$ there exists constants $C_{L,p}$ independent of n such that*

$$\|Q_n e^{-V_n}\|_{\mathbf{h},\hat{G}_{\kappa,\rho},\mathcal{A}_p,\delta_n} \leq C_{L,p} \bar{g}^{1/2} \tag{5.88}$$

$$\|Q_n e^{-V_n}\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A}_p,\delta_n} \leq C_{L,p} \bar{g}^{2} \tag{5.89}$$

Proof The bound (5.88) follows from (5.66) since $\hat{G}_{\kappa,\rho} > G_\kappa$. To prove (5.89) we first express $Q_n^{(m,m)}(X_{\delta_n}, \varphi + \zeta, \psi + \eta)$ in the Grassmann representation as in the proof of Lemma 5.10, substituting in the expressions there $\varphi \to \varphi + \zeta$, $\psi \to \psi + \eta$. Field derivatives are defined as in (5.26). For the bosonic coefficients we take derivatives at $\varphi = 0$. The resulting dependence on ζ is estimated by Lemma 5.2. The rest of the proof follows that of Lemma 5.10. We use Lemma 5.5 which implies that $\|e^{-V_n(X_{\delta_n},\zeta,\eta)}\|_{\mathbf{h}_*} \leq 2^{|X_{\delta_n}|}$, and X_{δ_n} is a small set. □

We now prove a lemma to control the perturbative flow coefficients a_n, b_n given in (4.15) and (4.17). This lemma is independent of the domain $\mathcal{D}_n$.

Lemma 5.12 *Let $v_{c*}^{(p)} = C_{c*,L}^p - C_{c*}^p$, with pointwise multiplication, where C_{c*} is the smooth continuum covariance in $\mathbb{R}^3$ of Corollary 1.1. Define*

$$a_{c*} = 2\int_{\mathbb{R}^3} dy\, v_{c*}^{(2)}(y), \qquad b_{c*} = 4\int_{\mathbb{R}^3} dy\, v_{c*}^{(3)}(y)$$

We have that a_n, b_n, a_{c*}, b_{c*} are strictly positive. Moreover there exist constants c_L independent of n such that

$$|a_n| \leq c_L, \qquad |b_n| \leq c_L, \qquad |a_{c*}| \leq c_L, \qquad |b_{c*}| \leq c_L \tag{5.90}$$

and

$$|a_n - a_{c*}| \leq c_L L^{-qn}, \qquad |b_n - b_{c*}| \leq c_L L^{-qn} \tag{5.91}$$

where $q > 0$ is as in Theorem 1.1.

Remark The convergence rate estimates (5.91) play no role in the estimates of the present section. They are used in Sect. 6 for the existence proof of a global renormalization group trajectory.

Proof From (4.16), for $p = 2, 3$, using $C_{n,L} = \Gamma_{n,L} + C_{n+1}$

$$v_{n+1}^{(p)} = C_{n,L}^{p} - C_{n+1}^{p} = \Gamma_{n,L}(\Gamma_{n,L}^{p-1} + p\Gamma_{n,L}^{p-2}C_{n+1} + \delta_{p,3}3C_{n+1}^{2}) \tag{5.92}$$

with pointwise multiplication. The positivity in Fourier space of the integral kernels on the right hand side implies that $a_n > 0$, $b_n > 0$ as claimed. The common factor of $\Gamma_{n,L}(x)$ which has finite range 1 implies that $v_{n+1}^{(p)}(x)$ has support in the unit ball in $(\delta_{n+1}\mathbb{Z})^3$. From Theorem 1.1 and Corollary 1.1 we have that v_{n+1}^{p} above are uniformly bounded in $L^\infty((\delta_{n+1}\mathbf{Z})^{\mathbf{3}})$ by constants c_L. By the same arguments v_{c*} has finite range and belongs to $L^\infty(\mathbb{R}^3)$. The uniform bounds in the first part of the lemma now follow.

By the same arguments using $C_{*,L} = \Gamma_{c*} + C_{c*}$ we have that $a_{c*} > 0$, $b_{c*} > 0$ and $v_{c*}^{(p)}(x)$ has support in the unit ball in $\mathbb{R}^3$. Moreover using Corollary 1.1 we have $\|v_{c*}^{(p)}\|_{C^k(\mathbb{R}^3)} \le c_{k,L}$ for all $\kappa \ge 0$.

Define

$$a_n^{(p)} = \int_{(\delta_{n+1}\mathbb{Z})^3} dy\, v_{n+1}^{(p)}(y), \qquad a_{c*}^{(p)} = \int_{\mathbb{R}^3} dy\, v_{c*}^{(p)}(y) \tag{5.93}$$

Then using the compact support property of $v_{n+1}^{(p)}$ and $v_{c*}^{(p)}$ we get

$$|a_n^{(p)} - a_{c*}^{(p)}| \le \|v_{n+1}^{(p)} - v_{c*}^{(p)}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} + \left|\int_{(\delta_{n+1}\mathbb{Z})^3} dy\, v_{c*}^{(p)}(y) - \int_{\mathbb{R}^3} dy\, v_{c*}^{(p)}(y)\right| \tag{5.94}$$

We estimate the first term on the right hand side of (5.94). We have

$$\|v_{n+1}^{(p)} - v_{c*}^{(p)}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} \le \|C_{c*,L}^{p} - C_{n,L}^{p}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} + \|C_{c*}^{p} - C_{n+1}^{p}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} \tag{5.95}$$

In the first term on the right in (5.95) we factor out $C_{c*,L} - C_{n,L}$ and in the second term we factor out $C_{c*} - C_{n+1}$. Then use of the bounds in Corollary 1.1 gives

$$\|v_{n+1}^{(p)} - v_{c*}^{(p)}\|_{L^\infty((\delta_{n+1}\mathbb{Z})^3)} \le c_{L,p}L^{-qn} \tag{5.96}$$

We estimate the second term on the right in (5.94) using Lemma 6.6 of [8] and the compact support of v_{c*}. This gives

$$\left|\int_{(\delta_{n+1}\mathbb{Z})^3} dy\, v_{c*}^{(p)}(y) - \int_{\mathbb{R}^3} dy\, v_{c*}^{(p)}(y)\right| \le O(1)\delta_{n+1}\|v_{c*}^{(p)}\|_{C^1(\mathbb{R}^3)} \le c_{L,p}L^{-(n+1)} \tag{5.97}$$

From (5.94), (5.96) and (5.97) we get with q that of Theorem 1.1

$$|a_n^{(p)} - a_{c*}^{(p)}| \le c_{L,p}L^{-qn} \tag{5.98}$$

which completes the proof of the lemma. □

Lemma 5.13 *Under the conditions of the domain $\mathcal{D}_n$ there exist constants $C_{p,L}$ independent of n such that*

$$\|Q_n(e^{-V_n} - e^{-\tilde{V}_n})\|_{\mathbf{h},\hat{G}_{\kappa,\rho},\mathcal{A}_p,\delta_n} \le C_{L,p}\bar{g}^{3/4} \tag{5.99}$$

$$\|Q_n(e^{-V_n}-e^{-\tilde{V}_n})\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A}_p,\delta_n} \le C_{L,p}\bar{g}^3 \tag{5.100}$$

Proof The proof is on the same lines as that of the corresponding Lemma 5.13 in [13]. It follows from Lemmas 5.11, 5.6, 5.10, and 5.5 which are lattice equivalents of the corresponding lemmas in [13]. □

Lemma 5.14 *Under the conditions of the domain $\mathcal{D}_n$ there exists constants C_L independent of n such that $K_n(\lambda)$ given by* (4.10) *satisfies the bounds*

$$\|K_n(\lambda)\|_{\mathbf{h},\hat{G}_{\kappa,\rho},\mathcal{A},\delta_n} \le C_L|\lambda\bar{g}^{1/4-\eta/3}|^2 \quad \text{for } |\lambda\bar{g}^{1/4-\eta/3}|<1 \tag{5.101}$$

$$\|K(\lambda)\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A},\delta_n} \le C_L|\lambda\bar{g}^{11/12-\eta/3}|^2 \quad \text{for } |\lambda\bar{g}^{11/12-\eta/3}|<1 \tag{5.102}$$

Proof This follows from Lemmas 5.11 and 5.13 and the hypothesis (5.5) on R_n. □

The following proposition shows how fluctuation integration of polymer activities passes through $\mathbf{h}$ and $\mathbf{h}_*$ norms. It will be put to use in the subsequent lemmas.

Lemma 5.14A

1. *Let $\tilde{K}(X_{\delta_n},\varphi,\zeta,\psi,\eta)$ be a polymer activity* (*see* (5.24) *and* (5.25)) *with norms defined as in* (5.26)–(5.32). *Let $\mathbf{h}=(h_B,h_F)$ and $\mathbf{h}_*=(h_{B*},h_F)$. Let $\tilde{K}^\sharp(X_{\delta_n},\varphi,\psi)=\int d\mu_{\Gamma_n}(\zeta)\,d\mu_{\Gamma_n}(\eta)\tilde{K}(X_{\delta_n},\varphi,\zeta,\psi,\eta)$. Then for h_F sufficiently large depending on L we have*

$$|\tilde{K}^\sharp(X_{\delta_n})|_{\mathbf{h}_*} \le \int d\mu_{\Gamma_n}(\zeta)\|\tilde{K}(X_{\delta_n},0,\zeta,0,0)\|_{\mathbf{h}_*} \tag{5.103}$$

$$\|\tilde{K}^\sharp(X_{\delta_n},\varphi,0)\|_{\mathbf{h}} \le \int d\mu_{\Gamma_n}(\zeta)\|\tilde{K}(X_{\delta_n},\varphi,\zeta,0,0)\|_{\mathbf{h}} \tag{5.104}$$

   where the norms on the left hand side are as in (2.16)–(2.18).
2. *Let $K(X_{\delta_n},\varphi,\psi)$ be a polymer activity in $\Omega^0(X_{\delta_n})$ and let $K^\sharp(X_{\delta_n},\varphi,\psi)=\int d\mu_{\Gamma_n}(\zeta)\,d\mu_{\Gamma_n}(\eta)K(X_{\delta_n},\varphi+\zeta,\psi+\eta)$. Let $\hat{\mathbf{h}}=(h_B,\frac{h_F}{2})$ and $\hat{\mathbf{h}}_*=(h_{B*},\frac{h_F}{2})$. Then for h_F sufficiently large depending on L we have*

$$|K^\sharp(X_{\delta_n})|_{\hat{\mathbf{h}}_*} \le \int d\mu_{\Gamma_n}(\zeta)\|K(X_{\delta_n},\zeta,0)\|_{\mathbf{h}_*} \tag{5.105}$$

$$\|K^\sharp(X_{\delta_n},\varphi,0)\|_{\hat{\mathbf{h}}} \le \int d\mu_{\Gamma_n}(\zeta)\|K(X_{\delta_n},\varphi+\zeta,0)\|_{\mathbf{h}} \tag{5.106}$$

   where the norms on both sides are as in (2.16)–(2.18).

Proof We get from the representation (5.25) and using the notations introduced there ((5.24), (5.25))

$$\begin{aligned}\tilde{K}^\sharp(X_{\delta_n},\varphi,\psi) = \int d\mu_{\Gamma_n}(\zeta)\Bigg[&\sum_{p\ge 0}\sum_{\substack{I\subset\{1,\dots,p\}\\ J\subset\{1,\dots,p\}}}\delta_{|I|,|J|}\frac{1}{|I|!\,|I^c|!\,|J|!\,|J^c|!}\\ &\times\int_{X_{\delta_n}^{2p}} d\mathbf{x}d\mathbf{y}\; D_F^{2p,IJ}\tilde{K}(X_{\delta_n},\varphi,\zeta,\mathbf{x}_I,\mathbf{x}_{I^c},\mathbf{y}_J,\mathbf{y}_{J^c})\\ &\times\psi(x_I)\bar{\psi}(y_J)\mathrm{det}_{\Gamma_n,I^c,J^c}(\mathbf{x}_{I^c},\mathbf{y}_{J^c})\times(-1)^\sharp\Bigg]\end{aligned} \tag{5.107}$$

Note that $|I^c| = |J^c|$ since $|I| = |J|$. The matrix $\det \Gamma_{n\, I^c, J^c}$ is an $|I^c| \times |I^c|$ square matrix whose entry $(\Gamma_{n\, I^c, J^c})_{rs}$ is given by

$$(\Gamma_{n\, I^c, J^c})_{rs} = \Gamma_n(x_r - y_s) \tag{5.108}$$

for $r \in I^c$, $s \in J^c$. $(-1)^\sharp$ is a sign factor which it is not necessary to specify. It may change from line to line.

We have

$$\begin{aligned}
&\frac{h_F^{2j}}{(j!)^2} D^{2j,m} \tilde{K}^\sharp(X_{\delta_n}, \varphi, 0, f^{\times m}, g_{2j}) \\
&\quad = \int d\mu_{\Gamma_n}(\zeta) \Bigg[\sum_{p \geq 0} \sum_{\substack{I \subset \{1,\ldots,p\} \\ J \subset \{1,\ldots,p\}}} \delta_{|I|,j} \delta_{|J|,j} \frac{h_F^{2|I|}}{|I|!\, |I^c|!\, |J|!\, |J^c|!} \\
&\qquad \times \int_{X_{\delta_n}^{2p}} d\mathbf{x} d\mathbf{y}\, D_B^m D_F^{2p,IJ} \tilde{K}(X_{\delta_n}, \varphi, \zeta, \mathbf{x}_I, \mathbf{x}_{I^c}, \mathbf{y}_J, \mathbf{y}_{J^c}; f^{\times m}) \\
&\qquad \times g_{2j}(\mathbf{x}_I, \mathbf{y}_J) \mathrm{det}_{\Gamma_n, I^c, J^c}(\mathbf{x}_{I^c}, \mathbf{y}_{J^c}) \times (-1)^\sharp \Bigg]
\end{aligned} \tag{5.109}$$

By definition $g_{2j}(\mathbf{x}_I, \mathbf{y}_J)$ (note that $|I| = |J| = j$), is antisymmetric in the members of $\mathbf{x}_I$ and in the members of $\mathbf{y}_J$. The determinant is antisymmetric in the members of $\mathbf{x}_{I^c}$ and in the members of $\mathbf{y}_{J^c}$.

Therefore the function

$$(g_{2j} \otimes \mathrm{det}_{\Gamma_n, I^c, J^c})(\mathbf{x}_I, \mathbf{x}_{I^c}, \mathbf{y}_J, \mathbf{y}_{J^c}) = g_{2j}(\mathbf{x}_I, \mathbf{y}_J) \mathrm{det}_{\Gamma_n, I^c, J^c}(\mathbf{x}_{I^c}, \mathbf{y}_{J^c}) \tag{5.110}$$

on $\tilde{X}_{\delta_n}^{|I|} \times \tilde{X}_{\delta_n}^{|I^c|} \times \tilde{X}_{\delta_n}^{|J|} \times \tilde{X}_{\delta_n}^{|J^c|} = \tilde{X}_{\delta_n}^{2p}$ is an admissable test function for the norm defined in (5.26), (5.27). Hence we get from (5.109) and (5.110)

$$\begin{aligned}
&\frac{h_F^{2j}}{(j!)^2} \left| D^{2j,m} \tilde{K}^\sharp(X_{\delta_n}, \varphi, 0, f^{\times m}, g_{2j}) \right| \\
&\quad \leq \int d\mu_{\Gamma_n}(\zeta) \Bigg[\sum_{p \geq 0} \sum_{\substack{I \subset \{1,\ldots,p\} \\ J \subset \{1,\ldots,p\}}} \delta_{|I|,j} \delta_{|J|,j} \frac{h_F^{2|I|}}{|I|!\, |I^c|!\, |J|!\, |J^c|!} \\
&\qquad \times \| D^{2p,IJ,m} \tilde{K}(X_{\delta_n}, \varphi, \zeta, , 0, 0) \| \prod_{j=1}^m \| f_j \|_{C^2(X_{\delta_n})} \\
&\qquad \times \| g_{2j} \otimes \mathrm{det}_{\Gamma_n, I^c, J^c} \|_{C^2(X_{\delta_n}^{2p})} \Bigg]
\end{aligned} \tag{5.111}$$

We have

$$\| g_{2j} \otimes \mathrm{det}_{\Gamma_n, I^c, J^c} \|_{C^2(X_{\delta_n}^{2p})} \leq \| g_{2j} \|_{C^2(X_{\delta_n}^{2j})} \| \mathrm{det}_{\Gamma_n, I^c, J^c} \|_{C^2(X_{\delta_n}^{2(p-j)})} \tag{5.112}$$

where $X_{\delta_n}^{2j} = X_{\delta_n}^{|I|} \times X_{\delta_n}^{|J|}$ and $X_{\delta_n}^{2(p-j)} = X_{\delta_n}^{|I^c|} \times X_{\delta_n}^{|J^c|}$, since $|I| = |J| = j$ and $|I^c| = |J^c| = p - j$.

Let $\partial_{\delta_n}^k$ be the lattice forward derivative of order k in multi-index notation. By part (5a) of Theorem 1.1 we have

$$\max_{0\le k\le 4} \|\partial_{\delta_n}^k \Gamma_n\|_{L^\infty((\delta_n\mathbb{Z})^3)} \le C_L \tag{5.113}$$

where C_L is a constant independent of n. Relabel $(\xi_1, \ldots, \xi_{2(p-j)}) = (x_1, \ldots, x_{|I^c|}, y_1, \ldots, y_{|J|^c})$ where the $x_i \in I^c$ and the $y_j \in J^c$. Let now $\partial_{\delta_n}^{k_r}$ be the forward lattice derivative of order k_r, $0 \le k_r \le 2$ with respect to the points ξ_r. Let $\mathbf{k} = (k_1, \ldots, k_{2(p-j)})$ and define $\partial_{\delta_n}^{\mathbf{k}} = \prod_{r=1}^{2(p-j)} \partial_{\delta_n}^{k_r}$. Let $\partial_{\delta_n}^{\mathbf{k}}$ act on the determinant. This produces another determinant with derivatives acting on the matrix elements $\Gamma_n(x_r - y_s)$. Since Γ_n is positive definite these matrices can be written as Gram matrices by a standard argument. Thus the matrix $a_{rs} = \partial_{\delta_n}^{k_r}\partial_{\delta_n}^{k_s}\Gamma_n(x_r - y_s)$ with $(x_r, y_s) \in X_{\delta_n} \times X_{\delta_n}$ can be written as $a_{rs} = (f_r, g_s)_{L^2((\delta_n\mathbb{Z})^3)}$ where $f_r(\cdot) = \partial_{\delta_n}^{k_r}\Gamma_n^{\frac12}(x_r, \cdot)$ and $g_s(\cdot) = \partial_{\delta_n}^{k_s}\Gamma_n^{\frac12}(\cdot, y_s)$. Gram's inequality says $|\det a_{rs}| \le \prod_{r=1}^{p-j}\|f_r\|_{L^2((\delta_n\mathbb{Z})^3)}\prod_{s=1}^{p-j}\|g_s\|_{L^2((\delta_n\mathbb{Z})^3)}$. We have $\|f_r\|^2_{L^2((\delta_n\mathbb{Z})^3)} = \partial_{\delta_n}^{k_r}\partial_{\delta_n}^{k_s}\Gamma_n(x_r - x_s)|_{r=s}$. Similarly, $\|g_s\|^2_{L^2((\delta_n\mathbb{Z})^3)} = \partial_{\delta_n}^{k_s}\partial_{\delta_n}^{k_s}\Gamma_n(y_r - y_s)|_{r=s}$. Since $2k_r \le 4$, we have by (5.113) the bound $\|f_r\|^2_{L^2((\delta_n\mathbb{Z})^3)} \le C_L$. Similarly $\|g_s\|^2_{L^2((\delta_n\mathbb{Z})^3)} \le C_L$. We therefore get

$$|\partial_{\delta_n}^{\mathbf{k}} \det \Gamma_{n\, I^c, J^c}(\mathbf{x}_{I^c}, \mathbf{y}_{J^c})| \le C_L^{p-j} \tag{5.114}$$

Hence

$$\|\det \Gamma_{n\, I^c, J^c}\|_{C^2(X_{\delta_n}^{2(p-j)})} \le C_L^{p-j} \tag{5.115}$$

From (5.114), (5.112) and (5.111) we get

$$\begin{aligned}
&\frac{h_F^{2j}}{(j!)^2}\|D^{2j,m}\tilde K^\sharp(X_{\delta_n}, \varphi, 0)\| \\
&\quad \le \int d\mu_{\Gamma_n}(\zeta)\Bigg[\sum_{p\ge 0}\sum_{\substack{I\subset\{1,\ldots,p\}\\ J\subset\{1,\ldots,p\}}} \delta_{|I|,j}\delta_{|J|,j}\left(\frac{C_L}{h_F^2}\right)^{p-j}\frac{h_F^{2p}}{|I|!\,|I^c|!\,|J|!\,|J^c|!} \\
&\qquad \times \|D^{2p,IJ,m}\tilde K(X_{\delta_n}, \varphi, \zeta, , 0, 0)\|\Bigg]
\end{aligned} \tag{5.116}$$

Now *choose h_F sufficiently large depending on L such that* $h_F^2 \ge C_L$ which implies that $(\frac{C_L}{h_F^2})^{p-j} \le 1$. Putting in this bound and then summing over j we get

$$\begin{aligned}
\sum_{j\ge 0}\frac{h_F^{2j}}{(j!)^2}\|D^{2j,m}\tilde K^\sharp(X_{\delta_n}, \varphi, 0)\| \le \int d\mu_{\Gamma_n}(\zeta)\Bigg[\sum_{p\ge 0}\sum_{\substack{I\subset\{1,\ldots,p\}\\ J\subset\{1,\ldots,p\}}} \delta_{|I|,|J|}\frac{h_F^{2p}}{|I|!\,|I^c|!\,|J|!\,|J^c|!} \\
\times \|D^{2p,IJ,m}\tilde K(X_{\delta_n}, \varphi, \zeta, , 0, 0)\|\Bigg]
\end{aligned} \tag{5.117}$$

Set $\varphi = 0$ in (5.117). Multiply both sides by $\frac{h_{B*}^m}{m!}$ and sum over m, $0 \le m \le m_0$. Majorize by dropping the constraint $\delta_{|I|,|J|}$ on the right hand side. This gives

$$|\tilde{K}^{\sharp}(X_{\delta_n})|_{\mathbf{h}_*} \le \int d\mu_{\Gamma_n}(\zeta)\ \|\tilde{K}(X_{\delta_n}, 0, \zeta, , 0, 0)\|_{\mathbf{h}_*}$$

On the other hand multiplying both sides of (5.117) by $\frac{h_B^m}{m!}$ and summing over m, $0 \le m \le m_0$ gives

$$\|\tilde{K}^{\sharp}(X_{\delta_n}, \varphi, 0)\|_{\mathbf{h}} \le \int d\mu_{\Gamma_n}(\zeta)\ \|\tilde{K}(X_{\delta_n}, \varphi, \zeta, , 0, 0)\|_{\mathbf{h}}$$

This proves the first part of the lemma.

We next turn to the second part of the Lemma. This is a consequence of the first part and Lemma 5.4A. Define $\tilde{K}(X_{\delta_n}, \varphi, \zeta, \psi, \eta) = K(X_{\delta_n}, \varphi + \zeta, \psi + \eta)$. Then from (5.104) of the first part and (5.34) of Lemma 5.4A we have

$$\begin{aligned}\|K^{\sharp}(X_{\delta_n}, \varphi, 0)\|_{\hat{\mathbf{h}}} = \|\tilde{K}^{\sharp}(X_{\delta_n}, \varphi, 0)\|_{\hat{\mathbf{h}}} &\le \int d\mu_{\Gamma_n}(\zeta)\|\tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0)\|_{\hat{\mathbf{h}}} \\ &\le \int d\mu_{\Gamma_n}(\zeta)\|K(X_{\delta_n}, \varphi + \zeta, 0)\|_{\mathbf{h}}\end{aligned}$$

which proves (5.106). Equation (5.105) follows similarly by using (5.103) of the first part followed by (5.35) of Lemma 5.4A. □

The following lemma generalizes Lemma 5.15 of [13] to the lattice and the additional presence of Grassmann fields. It will play a key role later in obtaining contractive estimates.

Lemma 5.15 *For any polymer activity* $\tilde{K}(X_{\delta_n}, \phi + \zeta, \psi + \eta)$:

$$\begin{aligned}\|\tilde{K}(X_{\delta_n}, \zeta, 0)\|_{\mathbf{h}_*} &\le O(1)\tilde{G}_{\rho,\kappa}(X_{\delta_n}, \zeta) \\ &\quad \times \left[|\tilde{K}(X_{\delta_n})|_{\mathbf{h}_*} + h_B^{-m_0} h_{B*}^{m_0}\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}, G_\kappa}\right] \end{aligned} \tag{5.118}$$

$$\begin{aligned}\|\tilde{K}(Y_{\delta_n}, \phi, 0)\|_{\mathbf{h}} &\le O(1) e^{\gamma\bar{g}\int_{Z_{\delta_n}\setminus Y_{\delta_n}} dy(\phi\bar{\phi}(y))^2} G_\kappa(Z_{\delta_n}, \phi) \\ &\quad \times \left[|\tilde{K}(Y_{\delta_n})|_{\mathbf{h}} + L^{-m_0 d_s}\|\tilde{K}(Y_{\delta_n})\|_{h_F, L^{d_s} h_B, G_\kappa}\right] \end{aligned} \tag{5.119}$$

$$|\tilde{K}^{\sharp}(X_{\delta_n})|_{\hat{\mathbf{h}}_*} \le O(1) 2^{|X_{\delta_n}|}\left[|\tilde{K}(X_{\delta_n})|_{\mathbf{h}_*} + h_B^{-m_0} h_{B*}^{m_0}\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}, G_\kappa}\right] \tag{5.120}$$

$$\|\tilde{K}^{\sharp}(X_{\delta_n})\|_{\hat{\mathbf{h}}, G_{2\kappa}} \le 2^{|X_{\delta_n}|}\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}, G_\kappa} \tag{5.121}$$

where $\tilde{G}_{\rho,\kappa}$ *is as defined in* (5.19), *and* $m_0 = 9$ *is the maximum number of derivatives appearing in the definition of Kernel and h norms. In* (5.119), $Y_{\delta_n}, Z_{\delta_n}, \gamma$ *are as described in Lemma* 5.1. *Moreover in the above norms* $\mathbf{h}_* = (h_F, h_{B*})$, *and* $\hat{\mathbf{h}}_* = (\frac{h_F}{2}, h_{B*})$ *where* $h_{B*} = (\rho\kappa)^{-1/2}$, $\mathbf{h} = (h_F, h_B)$, $\hat{\mathbf{h}} = (\frac{h_F}{2}, h_B)$, $h_B = c\bar{g}^{-1/4}$, $c = O(1)$ *very small, and* h_F *is taken to be sufficiently large depending on* L.

The superscript $\sharp$ stands for $d\mu_{\Gamma_n}(\zeta)$ integration. ρ is chosen as in Lemma 5.3, and κ as in Lemma 2.1. Note that we have that the constant $C(\rho, \kappa, j)$ appearing in Lemma 5.2 (this bounds ζ^j) satisfies

$$C(\rho, \kappa, j) = h_{B*}^j O(1)^j \tag{5.122}$$

Proof We will first prove (5.118) following the lines of the proof of Lemma 5.15 of [13] where the Grassman fields were absent. Recall from the definition in (5.30) that

$$\|\tilde{K}(X_{\delta_n},0,0,\zeta,0,0)\|_{\mathbf{h}_*}=\sum_{m=0}^{m_0}\frac{h_{B*}^m}{m!}A_m \tag{5.123}$$

where

$$A_m=\sum_{p\geq 0}h_F^{2p}\frac{h_{B*}^m}{m!}\|D^{2p,m}\tilde{K}(X_{\delta_n},0,0,\zeta,0,0)\| \tag{5.124}$$

First conside the case $m=m_0$. Then

$$A_{m_0}\leq h_{B*}^{m_0}h_B^{-m_0}\|\tilde{K}(X_{\delta_n}\|_{\mathbf{h},G_\kappa}\tilde{G}_{\rho,\kappa} \tag{5.125}$$

since $G_\kappa\leq\tilde{G}_{\rho,\kappa}$. Now let $m<m_0$.

We expand in ζ in Taylor series with remainder

$$\begin{aligned}(D^{2p,m}\tilde{K})(X_{\delta_n},\zeta,0;f^{\times m},g_{2n})={}&\sum_{j=0}^{m_0-m-1}\frac{1}{j!}(D^{2p,j+m}\tilde{K})(X_{\delta_n},0,0;f^{\times m},\zeta^{\times j},g_{2p})\\&+\frac{1}{(m_0-m-1)!}\int_0^1 ds(1-s)^{m_0-m-1}(D^{2p,m}\tilde{K})\\&\times(X_{\delta_n},s\zeta,0;f^{\times m},\zeta^{\times m_0-m},g_{2p})\end{aligned} \tag{5.126}$$

Therefore

$$\begin{aligned}\|(D^{2p,m}\tilde{K})(X_{\delta_n},\zeta,0)\|\leq{}&\sum_{j=0}^{m_0-m-1}\frac{1}{j!}\|(D^{2p,j+m}\tilde{K})(X_{\delta_n},0,0)\|\|\zeta\|^j_{C^2(X_{\delta_n})}\\&+\frac{1}{(m_0-m-1)!}\int_0^1 ds(1-s)^{m_0-m-1}\|\zeta\|^{m_0-m}_{C^2(X_{\delta_n})}\\&\times\|(D^{2p,m_0}\tilde{K})(X_{\delta_n},s\zeta,0)\|\end{aligned}$$

Hence

$$\begin{aligned}A_m\leq{}&\sum_{j=0}^{m_0-m-1}\frac{(j+m)!}{j!m!}h_{B*}^{-j}\|\zeta\|^j_{C^2(X_{\delta_n})}\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}_*}\\&+\frac{m_0!h_{B*}^{m_0}h_B^{-m_0}}{m!(m_0-m-1)!}\int_0^1 ds(1-s)^{m_0-m-1}h_{B*}^{-(m_0-m)}\|\zeta\|^{m_0-m}_{C^2(X_{\delta_n})}\\&\times\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h},G_\kappa}\tilde{G}_{\rho,\kappa}(X_{\delta_n},s\zeta)\end{aligned}$$

By Lemma 5.2, with ζ replaced by $\sqrt{1-s^2}\zeta$, and (5.122),

$$h_{B*}^{-j}\|\zeta\|^j_{C^2(X_{\delta_n})}\leq O(1)^j\tilde{G}_{\rho,\kappa}(X_{\delta_n},\sqrt{1-s^2}\,\zeta)\frac{1}{(1-s^2)^{j/2}} \tag{5.127}$$

where $0 \le s < 1$. With $s = 0$ this bound is applied to the terms in the sum over j. For the Taylor remainder term take $j = m_0 - m$ and note that $(1-s)^{(m_0-m)/2-1}$ is integrable since $m_0 > m$. Hence:

$$\begin{aligned} A_m &\le O(1)\tilde{G}_{\rho,\kappa}(X,\zeta) \\ &\quad \times \left[\sum_{j=0}^{m_0-m-1} \frac{(j+m)!}{j!m!} \|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}_*} + \frac{m_0! h_{B*}^{m_0} h_B^{-m_0}}{m!(m_0-m-1)!} \|\tilde{K}(X_{\delta_n})\|_{\mathbf{h},G_\kappa} \right] \\ &\le O(1)\tilde{G}_{\rho,\kappa}(X,\zeta)\Big[\|\tilde{K}(X_{\delta_n})\|_{\mathbf{h}_*} + h_{B*}^{m_0} h_B^{-m_0} \|\tilde{K}(X_{\delta_n})\|_{\mathbf{h},G_\kappa} \Big] \end{aligned} \tag{5.128}$$

Summing (5.128) over $0 \le m \le m_0 - 1$ and adding (5.125) proves (5.118).

Inequality (5.119) is also proved in the same way as (5.118). We are estimating the $\mathbf{h}$ norm which is given by (5.34) with $\mathbf{h}_*$ replaced by $\mathbf{h}$, h_B* by h_B and ζ by φ. We replace $\tilde{G}_{\rho,\kappa}$ by G_κ. Then (5.125) remains true with ζ replaced by φ and $\tilde{G}_{\rho,\kappa}$ replaced by G_k. Subsequently for $m < m_0$ we expand in Taylor series as above but now in φ. We do the norm estimate as above but now using Lemma 5.1 in place of Lemma 5.2. For ε sufficiently small depending on L we have $\bar{g}$ sufficiently small and therefore h_B^{-1} is sufficiently small. Hence $h_B^{-j} C \le O(1)$ where $C = \kappa^{-j/2} O(1)$ is the constant appearing in Lemma 5.1. In the Taylor remainder term we replace h_B by $L^{d_s} h_B$. which leads to the factor $L^{-m_0 d_s}$.

Finally note that the inequality (5.120) now follows on using (5.105) of Lemma 5.14A, followed by (5.118) and then Lemma 5.3. Equation (5.121) follows from (5.106) of Lemma 5.14A on using the stability of the large field regulator G_κ. □

The next lemma extends Lemma 5.16 of [13] to the case when Grassmann fields are also present.

Lemma 5.16 *For any $q > 0$, there exists constants c_L independent of n such that for L sufficiently large, ε sufficiently small and h_F sufficiently large depending on L,*

$$\|\mathcal{S}(\lambda, K_n)^\natural\|_{\mathbf{h},G_\kappa,\mathcal{A}_p,\delta_{n+1}} \le q \quad \textit{when } |\lambda \bar{g}^{1/4-\eta/3}| \le c_L \tag{5.129}$$

$$|\mathcal{S}(\lambda, K_n)^\natural|_{\mathbf{h}_*,\mathcal{A}_p,\delta_{n+1}} \le q \quad \textit{when } |\lambda \bar{g}^{11/12-\eta/3}| \le c_L \tag{5.130}$$

where $\mathbf{h}_* = (h_{B*}, h_F)$ and $\natural$ denotes integration with respect to $d\mu_{\Gamma_{n,L}}(\xi)$, $\Gamma_{n,L} = S_{L^{-1}}\Gamma_n$ being the rescaled fluctuation covariance.

When $R_n = 0$ we may set $\eta = 0$ in (5.129) and replace $\lambda\bar{g}^{11/12-\eta/3}$ by $\lambda\bar{g}^{1/2-\delta/2}$ in (5.130).

Proof We suppress the dependence on λ which plays a passive role in most of the following and make the dependence explicit towards the end when necessary. We will apply the first part of Lemma 5.14A to the reblocked polymer activity $\mathcal{B}K_n(LZ_{\delta_n}, S_L\Phi, \xi) = \mathcal{B}K_n(LZ_{\delta_n}, S_L\varphi, \zeta, S_L\psi, \eta)$ which is a functional of $\tilde{K}_n$ and P_n where $\tilde{K}_n(X_{\delta_n}, \varphi, \zeta, \psi, \eta) = K_n(X_{\delta_n}, \varphi+\zeta, \psi+\eta)$ (see the definition of reblocking in Sect. 3.1). Recall the definition of rescaled polymer activities and rescaled covariances given by (3.22) and (3.21) in the Appendix to Sect. 3. The rescaled, reblocked activities are defined by (3.25). We get by virtue of (5.104) and (5.103)

$$\|\mathcal{S}(K_n))^\natural(Z_{\delta_{n+1}}, \varphi, 0)\|_{\mathbf{h}} \le \int d\mu_{\Gamma_{n,L}}(\zeta) \|(S_L\mathcal{B}K_n)(Z_{\delta_n}, \varphi, \zeta, 0)\|_{\mathbf{h}} \tag{5.131}$$

$$|\mathcal{S}(K_n))^{\natural}(Z_{\delta_{n+1}})|_{\mathbf{h}_*} \leq \int d\mu_{\Gamma_{n,L}}(\zeta)\|(S_L\mathcal{B}K_n)(Z_{\delta_n},0,\zeta,0)\|_{\mathbf{h}_*} \tag{5.132}$$

We now prove (5.129) starting from (5.131). This follows the lines of the proof of Lemma 5.16 [13]. Unfortunately, in the latter proof a minor error crept in A. Abdesselam (private communication) and we take this opportunity to correct it. Inserting the definition (4.11) in (5.131) and using the multiplicative property of the $\mathbf{h}$ norm we get

$$\begin{aligned}
&\|(\mathcal{S}(K_n)^{\natural})(Z_{\delta_{n+1}},\varphi,0)\|_{\mathbf{h}}\\
&\leq \sum_{N+M\geq 1}\frac{1}{N!M!}\sum_{(X_{\delta_n,j}),(\Delta_{\delta_n,i})\to LZ_{\delta_{n+1}}}\|e^{-\tilde{V}_{n,L}(Z_{\delta_{n+1}}\setminus L^{-1}(\mathbf{X}_{\delta_{n+1}}\cup\Delta_{\delta_{\mathbf{n+1}}}),\varphi)}\|_{\mathbf{h}}\\
&\quad\times\int d\mu_{\Gamma_{n,L}}(\zeta)\prod_{j=1}^{N}\|\tilde{K}_{n,L}(L^{-1}X_{\delta_n,j},\varphi,\zeta,0)\|_{\mathbf{h}}\prod_{i=1}^{M}\|P_{n,L}(L^{-1}\Delta_{\delta_n,i},\varphi,\zeta,0)\|_{\mathbf{h}}
\end{aligned}$$

whence

$$\begin{aligned}
&\|(\mathcal{S}(K_n)^{\natural})(Z_{\delta_{n+1}},\varphi,0)\|_{\mathbf{h}}\\
&\leq 2^{|Z_{\delta_{n+1}}|}\sum_{N+M\geq 1}\frac{1}{N!M!}\times\sum_{(X_j),(\Delta_{\delta_n,i})\to LZ_{\delta_n}}\int d\mu_{\Gamma_n}(\zeta)\hat{G}_{\kappa,\rho}(\mathbf{X}_{\delta_n}\cup\Delta_{\delta_{\mathbf{n}}},S_L\varphi,\zeta)\\
&\quad\times\prod_{j=1}^{N}\|\tilde{K}_n(X_j)\|_{\mathbf{h}_L,\hat{G}_{\kappa,\rho}}\prod_{i=1}^{M}\|P_n(\Delta_{\delta_n,i})\|_{\mathbf{h}_L,\hat{G}_{\kappa,\rho}}
\end{aligned}\tag{5.133}$$

where $\mathbf{h}_L=(L^{-d_s}h_B,L^{-d_s}h_F)$, $\mathbf{X}_{\delta_n}=\bigcup X_{\delta_n,j}$, $\Delta_{\delta_n}=\bigcup\Delta_{\delta_n,i}$ and we have bounded $e^{-\tilde{V}_{n,L}}$ using Lemma 5.5 which continues to apply.

Lemma 5.4 bounds the ζ integral by

$$\begin{aligned}
2^{|\mathbf{X}_{\delta_n}\cup\Delta_{\delta_{\mathbf{n}}}|}G_{3\kappa}(\mathbf{X}_{\delta_n}\cup\Delta_{\delta_{\mathbf{n}}},S_L\varphi)&\leq 2^{|\mathbf{X}_{\delta_n}\cup\Delta_{\delta_{\mathbf{n}}}|}G_\kappa(L^{-1}(\mathbf{X}_{\delta_{n+1}}\cup\Delta_{\delta_{n+1}}),\phi)\\
&\leq\prod_{j=1}^{M}2^{|X_{\delta_n,j}|}\prod_{i=1}^{N}2^{|\Delta_{\delta_n,i}|}G_\kappa(Z_{\delta_{n+1}},\phi)
\end{aligned}$$

since $L^{-1}(\mathbf{X}_{\delta_{n+1}}\cup\Delta_{\delta_{\mathbf{n+1}}})\subset Z_{\delta_{n+1}}$. Moreover for L sufficiently large

$$\|\tilde{K}_n(X_j)\|_{\mathbf{h}_L,\hat{G}_{\kappa,\rho}}\leq\|\tilde{K}_n(X_j)\|_{\hat{\mathbf{h}},\hat{G}_{\kappa,\rho}}\leq\|K_n(X_j)\|_{\mathbf{h},\hat{G}_{\kappa,\rho}}$$

where we have used Lemma 5.4A, (5.34) in the last step. Therefore

$$\begin{aligned}
&\|(\mathcal{S}(K_n)^{\natural})(Z_{\delta_{n+1}})\|_{\mathbf{h},\mathbf{G}_\kappa}\\
&\leq 2^{|Z_{\delta_{n+1}}|}\sum_{N+M\geq 1}\frac{1}{N!M!}\sum_{(X_{\delta_n,j}),(\Delta_{\delta_n,i})\to LZ_{\delta_n}}\prod_{j=1}^{M}2^{|X_{\delta_n,j}|}\|K_n(X_j)\|_{\mathbf{h},\hat{G}_{\kappa,\rho}}\\
&\quad\times\prod_{i=1}^{N}2^{|\Delta_{\delta_n,i}|}\|P_n(\Delta_{\delta_n,i})\|_{\mathbf{h},\hat{G}_{\kappa,\rho}}
\end{aligned}$$

From this point on we proceed as in the proof of Lemma 5.16 [13], the only difference being is that now we are on a lattice. The condition on the sum over the polymers above implies that $Z_{\delta_n} = (\bigcup L^{-1}\bar{X}^L_{\delta_n,j}) \cup (\bigcup L^{-1}\bar{\Delta}^L_{\delta_n,i})$. This also implies that $Z_{\delta_{n+1}} = (\bigcup L^{-1}\bar{X}^L_{\delta_{n+1},j}) \cup (\bigcup L^{-1}\bar{\Delta}^L_{\delta_{n+1},i})$.

Multiply both sides by $\mathcal{A}_p(Z_{\delta_{n+1}})$ and observe on the right hand side

$$\begin{aligned}\mathcal{A}_{p+1}(Z_{\delta_{n+1}}) &\le \prod_{j=1}^{M}\mathcal{A}_{p+1}(L^{-1}\bar{X}_{j,\delta_{n+1}})\prod_{i=1}^{N}\mathcal{A}_{p+1}(L^{-1}\bar{\Delta}_{\delta_{n+1},i})\\ &\le O(1)^{N+M}\prod_{j=1}^{M}\mathcal{A}_{-2}(X_{j,\delta_{n+1}})\prod_{i=1}^{N}\mathcal{A}_{-2}(\Delta_{\delta_{n+1},i})\end{aligned}$$

where we have first used the fact that the L-closures of the polymers are connected by definition of the reblocking operation, then Lemma 2.2 together with $|X_{j,\delta_{n+1}}| = |X_{j,\delta_n}|$ and $|\Delta_{\delta_{n+1},i}| = |\Delta_{\delta_n,i}|$. The last observation follows from our definition of polymers in Sect. 1.3 and (1.79). Therefore

$$\begin{aligned}&\|(\mathcal{S}(K_n)^{\natural})(Z_{\delta_{n+1}})\|_{\mathbf{h},\mathbf{G}_\kappa}\mathcal{A}_p(Z_{\delta_{n+1}})\\ &\quad\le \sum_{N+M\ge 1}\frac{1}{N!M!}O(1)^{N+M}\sum_{(X_j),(\Delta_{\delta_n,i})\to LZ}\prod_{j=1}^{M}\|K_n(X_{j,\delta_n})\|_{\mathbf{h},\hat{G}_{\kappa,\rho}}\mathcal{A}_{-1}(X_{j,\delta_n})\\ &\qquad\times\prod_{i=1}^{N}\|P_n(\Delta_{\delta_n,i})\|_{\mathbf{h},\hat{G}_{\kappa,\rho}}\mathcal{A}_{-1}(\Delta_{\delta_n,i})\end{aligned}$$

Fix any $\Delta_{\delta_{n+1}}$ and sum over $Z_{\delta_{n+1}} \ni \Delta_{\delta_{n+1}}$. This fixes on the right hand side the sum over $Z_{\delta_n} \ni \Delta_{\delta_n}$ with Δ_{δ_n} fixed by restriction. The spanning tree argument of Lemma 7.1 of [15] controls the sums over $N, M, Z_{\delta_n}, (X_{j,\delta_n}), (\Delta_{i,\delta_n}) \to LZ_{\delta_n}$ with the result (we have now made explicit the dependence on λ)

$$\begin{aligned}&\|(\mathcal{S}(\lambda, K_n)^{\natural})\|_{\mathbf{h},\mathbf{G}_\kappa,\mathcal{A}_\mathbf{p},\delta_{\mathbf{n+1}}}\\ &\quad\le O(1)\sum_{N\ge 1}O(1)^N L^{3N}\Big(\|K_n(\lambda)\|_{\mathbf{h},\hat{G}_{\kappa,\rho},\mathcal{A},\delta_n} + \|P_n(\lambda)\|_{\mathbf{h},\hat{G}_{\kappa,\rho},\mathcal{A},\delta_n}\Big)^N\end{aligned}$$

The proof of (5.129) is completed by Lemmas 5.8 and 5.14. When $R = 0$ we can use Lemma 5.8 and replace Lemma 5.14 by Lemma 5.11.

To prove (5.130) we start from (5.132) and proceed as before. We replace $\hat{G}_{\kappa,\rho}$ by $\tilde{G}_{\kappa,\rho}$ and then use Lemma 5.3 to estimate the ζ integral. We use (5.35) of Lemma 5.4A. Proceeding as before now leads to

$$\begin{aligned}&|(\mathcal{S}(\lambda, K_n)^{\natural})|_{\mathbf{h}_*,\mathcal{A}_p,\delta_{n+1}}\\ &\quad\le O(1)\sum_{N\ge 1}O(1)^N L^{3N}\Big(\|K_n(\lambda)\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A},\delta_n} + \|P(\lambda)\|_{\mathbf{h}_*,\tilde{G}_{\kappa,\rho},\mathcal{A},\delta_n}\Big)^N\end{aligned}$$

Now use Lemmas 5.8 and 5.14 to complete the proof of (5.130). Finally when $R = 0$ use Lemmas 5.8 and 5.11 as before. □

Estimates on Relevant Parts and Flow Coefficients from the Remainder

Let $(\tilde{\alpha}_{n,P})$ be the coefficients $(\tilde{\alpha}_{n,2,0}, \tilde{\alpha}_{n,2,1}, \tilde{\alpha}_{n,2,\bar{1}}\tilde{\alpha}_{n,4})$ defined in (4.43) and (4.61). The flow coefficients $\xi_{n,R}$, $\rho_{n,R}$ are given in (4.53).

Lemma 5.17 *Under the conditions of the domain $\mathcal{D}_n$ we have*

$$\|R_n^\sharp\|_{\hat{\mathbf{h}},G_{3\kappa},\mathcal{A}_{-1},\delta_n} \le \bar{g}^{3/4-\eta} \tag{5.134}$$

$$|R_n^\sharp|_{\hat{\mathbf{h}}_*,\mathcal{A}_{-1},\delta_n} \le O(1)\bar{g}^{11/4-\eta} \tag{5.135}$$

$$|\tilde{\alpha}_{n,P}|_{\mathcal{A},\delta_n} \le O(1)\bar{g}^{11/4-\eta} \tag{5.136}$$

$$|\xi_n| \le C_L\bar{g}^{11/4-\eta} \tag{5.137}$$

$$|\rho_n| \le C_L\bar{g}^{11/4-\eta} \tag{5.138}$$

where the constants C_L are independent of n and ε

Proof Equation (5.134) follows from (5.7) and Lemma 5.15, (5.121). Equation (5.135) follow from (5.8) and Lemma 5.15 with $m_0 = 9$ and ε sufficiently small depending on L so that $\bar{g}$ is sufficiently small. In fact in Lemma 5.15 (with $\tilde{K} = R_n$) the first term has the desired bound by (5.8). By (5.7) together with $h_B^{-1} = c\bar{g}^{\frac{1}{4}}$ and $h_{B*} = h_{B*}(L)$ we see that the second term is bounded by $O(1)\bar{g}^{\frac{1}{4}}h_{B*}^9\bar{g}^{11/4-\eta} \le \bar{g}^{11/4-\eta}$ for $\bar{g}$ sufficiently small.

Recall that $\tilde{\alpha}_{n,P}(X_{\delta_n})$, are supported on small sets. Then (5.136) follows from (4.61) and (5.135). In fact the dominant contribution comes by setting $\tilde{V}_n = 0$ because the difference gives additional powers of $\bar{g}$. Then we have

$$\|\tilde{\alpha}_{n,P}\|_{\mathcal{A},\delta_n} \le O(1)n(P)!\hat{\mathbf{h}}_*^{-n(P)}|1_S R_n^\sharp|_{\hat{\mathbf{h}}_*,\mathcal{A},\delta_n}$$

where $n(P)$ is the number of fields in the monomial P, we have used the shorthand notation $\hat{\mathbf{h}}_*^{-n(P)} = \max_{n(P)_F+n(P)_B=n(P)}(h_{*B}^{-n(P)_B}\hat{h}_F^{-n(P)_F})$ and 1_S is the indicator function on small sets. Now use (5.135) to get (5.136). Equations (5.137), (5.138) follow from (5.136), the definitions (4.53), (4.51) and Wick coefficients $C_n(0)$ are uniformly bounded by a L dependent constant by Corollary 1.1. □

Lemma 5.18 *Under the conditions of $\mathcal{D}_n$ and ε sufficiently small depending on L, there exists a constant C_L independent of ε and n such that*

$$|g_{n+1} - \bar{g}| < 2\nu\bar{g}^{3/2}, \qquad |\mu_{n+1}| < C_L\bar{g}^{2-\delta} \tag{5.139}$$

Proof It is convenient to define

$$\tilde{g}_n = g_n - \bar{g}$$

Then from the flow equation (4.39) for g_n and the definition of $\bar{g}$ in (5.2) we get

$$\tilde{g}_{n+1} = (2 - L^\varepsilon)\tilde{g}_n + \tilde{\xi}_n \tag{5.140}$$

where

$$\tilde{\xi}_n = -L^{2\varepsilon}a_{c*}\tilde{g}_n^2 - L^{2\varepsilon}(a_n - a_{c*})g_n^2 + \xi_n \tag{5.141}$$

From Lemmas 5.12, 5.17 and $g_n \in \mathcal{D}_n$ we get for ε sufficiently small depending on L the bound

$$|\tilde{\xi}_n| \le C_L \bar{g}^2 \tag{5.142}$$

Therefore

$$|\tilde{g}_{n+1}| < \nu \bar{g}\left((2 - L^{\varepsilon}) + \frac{C_L}{\nu}\bar{g}\right) \tag{5.143}$$

For ε sufficiently small depending on L we get

$$\left|(1 - L^{\varepsilon}) + \frac{C_L}{\nu}\bar{g}\right| \le 1 \tag{5.144}$$

Therefore $|\tilde{g}_{n+1}| \le 2\nu\bar{g}$ which proves the first inequality of (5.139).

The bound on μ_{n+1} follows from the second of the flow equations (4.15), on using μ_n, g_n belong to $\mathcal{D}_n$, Lemma 5.12 and the bound (5.138) on ρ_n. □

As stated in Theorem 3.1 borrowed from [6] the assumption of stability of the local potential with respect to perturbation by relevant parts (see (3.18) ensures the extraction estimate of (3.19). The following lemma proves the stability for the case at hand, namely that of $\tilde{V}_{n,L}(\Delta_{n+1})$ with respect to the relevant part F_n defined in Sect. 4.

Recall from (4.12) that $F_n(\lambda) = \lambda^2 F_{Q_n} + \lambda^3 F_{R_n}$ and from (3.15) that (each part of) F_n decomposes: $F_n(X_{\delta_{n+1}}) = \sum_{\Delta_{\delta_{n+1}} \subset X_{\delta_{n+1}}} F_n(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}})$.

Lemma 5.19 *For any $R > 0$ and $\xi := R\max(|\lambda^2|\bar{g}, |\lambda^3|\bar{g}^{7/4-\eta})$ sufficiently small,*

$$\|e^{-\tilde{V}_{n,L}(\Delta_{n+1}) - \sum_{X_{\delta_{n+1}} \supset \Delta_{n+1}} z(X_{\delta_{n+1}})F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})}\|_{\mathbf{h}, G_\kappa} \le 2^2 \tag{5.145}$$

where $z(X_{\delta_{n+1}})$ are complex parameters with $|z(X_{\delta_{n+1}})| \le R$.

Proof It is easy to see that Lemma 5.5 still holds if we replace $\tilde{V}_n$ by $\tilde{V}_{n,L}$ provided ε is sufficiently small. This implies that $\bar{g}$ is sufficiently small. We then have

$$\begin{aligned} &\|e^{-\tilde{V}_{n,L}(\Delta_{n+1}) - \sum_{X_{\delta_{n+1}} \supset \Delta_{n+1}} z(X_{\delta_{n+1}})F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})}\|_{\mathbf{h}} \\ &\quad \le 2\, e^{-\bar{g}/4 \int_{\Delta_{n+1}} dx (|\varphi(x)|^2)^2 + \sum_{X_{\delta_{n+1}} \supset \Delta_{n+1}} R\|F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}}} \end{aligned} \tag{5.146}$$

Recall that the relevant parts $F(X_{\delta_{n+1}}, \Delta_{n+1})$ are supported on small sets $X_{\delta_{n+1}}$. The proof now follows easily from the following □

Claim *For ε sufficiently small*

$$\begin{aligned} &|z(X_{\delta_{n+1}})| \|F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} \\ &\quad \le C_L \xi\left(\bar{g}\int_{\Delta_{n+1}} d^3x (|\varphi(x)|^2)^2 + \bar{g}^{1/2} \||\varphi|^2\|_{\Delta_{n+1},1,5} + 1\right) \end{aligned} \tag{5.147}$$

where $\|\phi\|^2_{\Delta_{n+1},1,5}$ is the square of the lattice Sobolev norm defined in (2.1).

Proof of the Claim We have $\|F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} \le |\lambda|^2 \|F_Q(X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} + |\lambda|^3 \times \|F_R(X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}}$.

Consider (4.27)–(4.31). Undo the Wick ordering on the superfield field and note that, by virtue of supersymmetry, no field independent terms arise. The $m = 1$ term in (4.30) remains unchanged and in the $m = 2$ case there results an additional contribution $-2C_{n+1}(0)\Phi\bar{\Phi}$. The Wick constant $C_{n+1}(0)$ has a uniform bound C which depends only on L by Corollary 1.1. We write this in the Grassmann representation and notice that for the $m = 2$ case the $\psi\bar{\psi}(x)^2$ contribution vanishes by statistics. From the definition of the $\mathbf{h}$ norm with $h_B = c\bar{g}^{-1/4}$ and $h_F = h_F(L)$ we get the bound

$$\begin{aligned}
&\|F_Q(X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} \\
&\quad\le C_L\bar{g}^2\Bigg(\int_{\Delta_{n+1}} d^3x \|(|\varphi|^2)^2(x)\|_{h_B} \sup_{x\in\Delta_{n+1}} |f_Q^{(2)}(X_{\delta_{n+1}}, x, \Delta_{n+1})| \\
&\qquad + \left(\int_{\Delta_{n+1}} d^3x \|(|\varphi|^2)(x)\|_{h_B} + 1\right)\sum_{m=1}^{2} \sup_{x\in\Delta_{n+1}} |f_Q^{(m)}(X_{\delta_{n+1}}, x, \Delta_{n+1})|\Bigg)
\end{aligned}$$

Now for $m = 1, 2$

$$\bar{g}\int_{\Delta_{n+1}} d^3x \|(|\varphi|^2)^m(x)\|_{h_B} \le O(1)\left(\bar{g}\int_{\Delta_{n+1}} d^3x (|\varphi|^2)^2(x) + 1\right)$$

From the definition (4.31) and the estimates obtained in the course of proving Lemma 5.12, we have

$$\sup_{x\in\Delta_{n+1}} |f_Q^{(m)}(X_{\delta_{n+1}}, x, \Delta_{n+1})| \le C_L$$

Therefore

$$|\lambda^2||z(X_{\delta_{n+1}})|\ \|F_Q(X_{\delta_{n+1}}, \Delta_{n+1})\|_h \le C_L R|\lambda|^2\bar{g}\left(\bar{g}\int_{\Delta_{n+1}} dx (|\varphi|^2)^2(x) + 1\right) \tag{5.148}$$

Next consider F_{R_n}, supported on small sets, defined in (4.43), (4.46). Recall (4.48),

$$F_R(X_{\delta_{n+1}}, \Phi) = \sum_P \int_{\Delta_{n+1}} dx\ \alpha_P(X_{\delta_{n+1}}, x) P(\Phi(x), \partial_{\delta_{n+1}}\Phi(x))$$

By Lemma 5.17 and (4.49) we have $|\alpha_P(X_{\delta_{n+1}}, x)| \le C_L\bar{g}^{11/4-\eta}$, so that

$$\begin{aligned}
&|\lambda|^3|z(X_{\delta_{n+1}})|\|F_R(X_{\delta_{n+1}}, \Delta_{n+1})\|_{\mathbf{h}} \\
&\quad\le C_L R|\lambda|^3\bar{g}^{11/4-\eta}\sum_P \int_{\Delta_{n+1}} dx\ \|P(\Phi(x), \partial_{\delta_{n+1}}\Phi(x))\|_{\mathbf{h}} \\
&\quad\le C_L R|\lambda|^3\bar{g}^{7/4-\eta}\left(\bar{g}\int_{\Delta_{n+1}} dx\ (|\varphi|^2)^2(x) + \bar{g}^{1/2}\||\varphi|^2\|_{\Delta_{n+1},1,5} + 1\right)
\end{aligned}$$

The claim follows by combining this with (5.147). In the above inequality the Sobolev norm when estimating the term giving arise to $\phi\partial_{\delta_n,\mu}\bar{\phi}$. We bound $|\phi\partial_{\delta_n,\mu}\bar{\phi}| \le 1/2(||\varphi|^2| + |\partial_{\delta_n,\mu}\phi|^2)$ and then use the lattice Sobolev embedding inequality. □

Lemma 5.20 *For any* $R > 0$ *and* $\xi := R\max(|\lambda^2|\bar{g}^2, |\lambda^3|\bar{g}^{11/4-\eta})$ *sufficiently small*,

$$|e^{-\tilde{V}_{n,L}(\Delta_{n+1})-\sum_{X_{\delta_{n+1}}\supset\Delta_{n+1}} z(X_{\delta_{n+1}})F(\lambda,X_{\delta_{n+1}},\Delta_{n+1})}|_{\mathbf{h}_*} \leq 2^2 \tag{5.149}$$

where $z(X)$ *are complex parameters with* $|z(X)| \leq R$.

Proof The proof is similar to the previous one except that we can use the estimate $|F(\lambda, X, \Delta)|_{\mathbf{h}_*} \leq C_L\xi$ in place of (5.147) since the $\mathbf{h}_*$ norm is computed with field derivatives at $\Phi = 0$. □

We will now bound the remainder R_{n+1} given in (4.41). It consist of a sum of four contributions, namely $R_{n+1,\text{main}}$, $R_{n+1,\text{linear}}$, $R_{n+1,3}$ and $R_{n+1,4}$ which we will estimate in turn. These estimates parallel those obtained for the continuum bosonic theory in [13].

Recall from (4.36) that

$$R_{n+1,\text{main}} = \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4}\mathcal{E}\Big(\mathcal{S}(\lambda, Q_n e^{-V_n})^\natural, F_{Q_n}(\lambda)\Big) \tag{5.150}$$

Lemma 5.21

$$\|R_{n+1,\text{main}}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \leq C_L\bar{g}^{3/4} \tag{5.151}$$

$$|R_{n+1,\text{main}}|_{\mathbf{h}_*,\mathcal{A},\delta_{n+1}} \leq C_L\bar{g}^{3-3\delta/2} \tag{5.152}$$

Proof The proof is identical to that of Lemma 5.21 of [13] except that we replace ε by $\bar{g}$, put in lattice subscript n where appropriate, and note that the field independant piece F_0 is now absent. We apply Theorem 3.1 (which is a restatement of Theorem 5 in Sect. 4.2 of [6]) instead of Theorem 6 of [6]. □

Recall from (4.38) that

$$R_{n+1,3} = \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4(\lambda-1)}\mathcal{E}\Big(\mathcal{S}(\lambda, K_n)^\natural, F_n(\lambda)\Big) \tag{5.153}$$

Lemma 5.22

$$\|R_{n+1,3}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \leq C_L\bar{g}^{1-4\eta/3} \tag{5.154}$$

$$|R_{n+1,3}|_{\mathbf{h}_*,\mathcal{A},\delta_{n+1}} \leq C_L\bar{g}^{10/3-4\eta/3} \tag{5.155}$$

Proof The proof follows the lines of that of Lemma 5.21. To prove (5.154) we take the contour γ to of radius $|\lambda| = c_L\bar{g}^{-(1/4-\eta/3)}$. This ensures that the hypothesis of Lemma 5.16, (5.130) is satisfied and ξ of Lemma 5.19 is sufficiently small so that stability holds. Equation (5.154) now follows from the extraction estimate (3.19) and the Cauchy bound as before. To prove (5.154) we take γ to be of radius $|\lambda| = c_L\bar{g}^{-(5/6-\eta/3)}$. Then for $\bar{g}$ sufficiently small the hypothesis for (5.130) of Lemma 5.16 is satisfied. Moreover then ξ of Lemma 5.20 is sufficiently small and Lemma 5.20 holds. Equation (5.154) now follows from the extraction estimate (3.20) and the Cauchy bound as before. □

From the definition of $R_{n+1,4}$ in (4.40) we have

$$R_{n+1,4} = \left(e^{-V_{n+1}} - e^{-\tilde{V}_{n,L}}\right) Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n+1}) + e^{-\tilde{V}_{n,L}} Q(C_{n+1}, \mathbf{w}_{n+1}, (g_{n+1}^2 - g_{n,L}^2))$$

Lemma 5.23

$$\|R_{n+1,4}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \le C_L \bar{g}^{3/2}, \qquad |R_{n+1,4}|_{\mathbf{h}_*,\mathcal{A},\delta_{n+1}} \le C_L \bar{g}^3$$

Proof The proof is the same as that of Lemma 5.23 of [13] except that we replace ε by $\bar{g}$, insert lattice subscript n as appropriate and use the lattice counterparts (that we have already established) of the lemmas exploited in [13] for the proof. □

Lemma 5.24 *Let X_{δ_n} be a small set and let $J(X_{\delta_n}, \Phi)$ be normalized as in (4.59). Recall that the rescaled activity J_L is defined by $J_L(L^{-1}X_{\delta_{n+1}}, \varphi, \psi) = J(X_{\delta_n}, \varphi_{L^{-1}}, \psi_{L^{-1}})$ where $\varphi_{L^{-1}} = S_L\varphi$ and $\psi_{L^{-1}} = S_L\psi$. Then we have*

1. *For $2p + m = 2$ so that $(p, m) = (1, 0),\ (0, 2)$*

$$\|D^{2p,m} J_L(L^{-1}X_{\delta_{n+1}}, 0, 0)\| \le O(1) L^{-(7-\varepsilon)/2} \|D^{2p,m} J(X_{\delta_n}, 0, 0)\| \tag{5.156}$$

2. *For $2p + m = 4$ so that $(p, m) = (2, 0), (1, 2), (0, 4)$*

$$\|D^{2p,m} J_L(L^{-1}X_{\delta_{n+1}}, 0, 0)\| \le O(1) L^{-(4-\varepsilon)} \|D^{2p,m} J(X_{\delta_n}, 0, 0)\| \tag{5.157}$$

Proof In the proof of this lemma we will need to use the lattice Taylor expansion introduced in (5.13), (5.14) and (5.15) with a particular choice of a lattice path joining two points. The polymer X_{δ_n} being a small set is connected. It can be represented as $X_{\delta_n} = X \cap (\delta_n\mathbb{Z})^3$ where X is a continuum connected polymer which is a small set. By the argument in the proof of Lemma 5.1 it suffices to consider the case when X_{δ_n} is a block. Then the lattice path lies entirely in X_{δ_n}.

Let u_k be a function defined on $(\tilde{X}_{\delta_n}^{(2)})^k$ for $k \ge 2$. In the following u is one of the test functions of Sect. 2.2. Thus u will represent either one of the functions f_j defined on $\tilde{X}_{\delta_n}^{(2)}$ giving a direction for a bosonic derivative or a function g_{2p} defined on $(\tilde{X}_{\delta_n}^{(2)})^{2p} = (\tilde{X}_{\delta_n}^{(2)})^p \times (\tilde{X}_{\delta_n}^{(2)})^p$ associated with a fermionic derivative of order $2p$. Note that g_{2p} is restricted to be antisymmetric in the sense explained in the lines preceding equation (2.12). The $C^2(X_{\delta_n}^k)$ norms of these functions for $k = 1,\ 2p$ are defined as in (2.14) and (2.15) of Sect. 2.2.

We recall the definition of the rescaled function

$$S_L u_k(x) = u_{k,L^{-1}}(x) = L^{-kd_s} u_k\left(\frac{x}{L}\right)$$

Observe that

$$\|u_{k,L^{-1}}\|_{C^2(X_{\delta_n}^k)} \le L^{-kd_s} \|u_k\|_{C^2(L^{-1}X_{\delta_{n+1}}^k)} \tag{5.158}$$

Let $e_1, e_2, \ldots, e_{3k}$ be the basis vectors of $(\delta_n\mathbb{Z})^{3k}$. Let $x = \sum_{i=1}^{3k}(x, e_i)e_i$ denote a point in $X_{\delta_n}^k$. Fix a point $x_0 \in X_{\delta_n}^k$. We write $x - x_0 = \delta_n \sum_{i=1}^{3k} h_i \varepsilon_i e_i$ where h_i are non-negative integers and $\varepsilon_i = \mathrm{sign}(x - x_0)_i$.

From (5.15) we have

$$u_{k,L^{-1}}(x) = u_{k,L^{-1}}(x_0) + \sum_{i=1}^{3k}((x-x_0), e_i)\partial_{\delta_n,\varepsilon_i e_i} u_{k,L^{-1}}(x_0)$$
$$+ \delta_n^2 \sum_{i,j=1}^{3k}\sum_{s_i=0}^{h_i-1}\sum_{s_j=0}^{h_j-1} \partial_{\delta_n,\varepsilon_i e_i}\partial_{\delta_n,\varepsilon_j e_j} u_{k,L^{-1}}(x_0 + p_j(p_i(x-x_0,s_i),s_j))$$

The argument of $u_{k,L^{-1}}$ in the last term lies entirely in $X^k_{\delta_n}$ since X_{δ_n} is a block.

Define

$$\delta u_{k,L^{-1}}(x) = u_{k,L^{-1}}(x) - u_{k,L^{-1}}(x_0)$$
$$= \delta_n \sum_{j=1}^{3k}\sum_{s_j=0}^{h_j-1} \partial_{\delta_n,\varepsilon_j e_j} u_{k,L^{-1}}(x_0 + p_j(x-x_0,s_j)) \tag{5.159}$$

and

$$\delta^2 u_{k,L^{-1}}(x) = h_{k,L^{-1}}(x) - u_{k,L^{-1}}(x_0) - \sum_{i=1}^{3k}((x-x_0), e_i)\partial_{\delta_n,\varepsilon_i e_i} u_{k,L^{-1}}(x_0)$$
$$= \delta_n^2 \sum_{i,j=1}^{3k}\sum_{s_i=0}^{h_i-1}\sum_{s_j=0}^{h_j-1} \partial_{\delta_n,\varepsilon_i e_i}\partial_{\delta_n,\varepsilon_j e_j} u_{k,L^{-1}}(x_0 + p_j(p_i(x-x_0,s_i),s_j)) \tag{5.160}$$

Now from (5.159), (5.160) we have using the definition of the rescaled function $u_{k,L^{-1}}$

$$\delta u_{k,L^{-1}}(x) = L^{-(1+k\frac{3-\varepsilon}{4})}\delta_n \sum_{j=1}^{3k}\sum_{s_j=0}^{h_j-1} (\partial_{\delta_n,\varepsilon_j e_j} u_k)(L^{-1}(x_0 + p_j(x-x_0,s_j)))$$

and for $l \ge 1$

$$\partial^l_{\delta_n}\delta u_{k,L^{-1}}(x) = L^{-(l+k\frac{3-\varepsilon}{4})}(\partial^l_{\delta_n} u_k)(L^{-1}x)$$

where for $\partial^l_{\delta_n}$ a multi-index convention is implicit. This implies that

$$\|\delta u_{k,L^{-1}}\|_{C^2(X^k_{\delta_n})} \le c_1 L^{-(1+k\frac{3-\varepsilon}{4})}\|u_k\|_{C^2(L^{-1}X^k_{\delta_{n+1}})} \tag{5.161}$$

where $c_1 = O(1)$ since X is a small set. In the same way starting from (5.160) a little bit of work shows that

$$\|\delta^2 u_{k,L^{-1}}\|_{C^2(X^k_{\delta_n})} \le c_2 L^{-(2+k\frac{3-\varepsilon}{4})}\|u_k\|_{C^2(L^{-1}X^k_{\delta_{n+1}})} \tag{5.162}$$

where $c_2 = O(1)$.

We will first prove the bounds of (5.156).

Consider first the case $(p,m) = (1,0)$. We have

$$\begin{aligned} D^{2,0} J_L(L^{-1} X_{\delta_{n+1}}, 0, 0; g_2) &= D^{2,0} J(X_{\delta_n}, 0, 0; g_{2,L^{-1}}) \\ &= D^{2,0} J(X_{\delta_n}, 0, 0; \delta^2 g_{2,L^{-1}}) \end{aligned} \tag{5.163}$$

where we have Taylor expanded the function $g_{2,L^{-1}}$ as in (5.160) and then used the first and second normalization conditions in (4.59). Therefore

$$\begin{aligned} |D^{2,0} J_L(L^{-1} X_{\delta_{n+1}}, 0, 0; g_2)| &\le \|D^{2,0} J(X_{\delta_n}, , 0, 0)\| \, \|\delta^2 g_{2,L^{-1}}\|_{C^2(X^2_{\delta_n})} \\ &\le O(1) L^{-(7-\varepsilon)/2} \|D^{2,0} J(X_{\delta_n}, 0)\| \, \|g_2\|_{C^2(L^{-1} X^2_{\delta_{n+1}})} \end{aligned}$$

where in the last step we have used the bound in (5.162) for $k = 2$. This proves the case $(p, m) = (1, 0)$ of the lemma.

To prove the case $(p, m) = (0, 2)$ we Taylor expand the function $f_{L^{-1}}(x)$ to second order, then use the first and the third conditions in (4.59) to get

$$\begin{aligned} D^{0,2} J(X_{\delta_n}, 0, 0; f^{\times 2}_{L^{-1}}) &= D^{0,2} J(X_{\delta_n}, 0, 0; f_{1,L^{-1}}(x_0), \delta^2 f_{2,L^{-1}}) \\ &\quad + D^{0,2} J(X_{\delta_n}, 0, 0; \delta^2 f_{1,L^{-1}}, f_{2,L^{-1}}(x_0)) \\ &\quad + D^{0,2} J(X_{\delta_n}, 0, 0; \delta f_{1,L^{-1}}, \delta f_{2,L^{-1}}) \end{aligned} \tag{5.164}$$

Therefore

$$\begin{aligned} |D^{0,2} J(X_{\delta_n}, 0, 0; f^{\times 2}_{L^{-1}})| &\le \|D^{0,2} J(X_{\delta_n}, 0, 0)\| \Big(\|f_{1,L^{-1}}\|_{C^2(X_{\delta_n})} \|\delta^2 f_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \\ &\quad + \|f_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \|\delta^2 f_{1,L^{-1}}\|_{C^2(X_{\delta_n})} \\ &\quad + \|\delta f_{1,L^{-1}}\|_{C^2(X_{\delta_n})} \|\delta f_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \Big) \\ &\le O(1) L^{-(7-\varepsilon)/2} \|D^{0,2} J(X_{\delta_n}, 0, 0)\| \prod_{j=1}^{2} \|f_j\|_{C^2(L^{-1} X_{\delta_{n+1}})} \end{aligned}$$

where we have used the bounds (5.158), (5.161) and (5.162) for the case $k = 1$. This proves the case $(p, m) = (0, 2)$.

Next we prove the bounds (5.157). For this case $(p, m) = (2, 0), (1, 2), (0, 4)$ so that $2p + m = 4$. Taylor expand test functions around the fixed point $x_0 \in X_{\delta_n}$ to first order with remainder. We get for $(p, m) = (2, 0)$

$$D^{4,0} J_L(L^{-1} X_{\delta_{n+1}}, 0; g_4) = D^{4,0} J(X_{\delta_n}, 0; g_{4,L^{-1}}) = D^{4,0} J(X_{\delta_n}, 0; \delta g_{4,L^{-1}}) \tag{5.165}$$

where we have Taylor expanded the function $g_{4,L^{-1}}$ as in (5.159) and then used (4.58).

Therefore exploiting the bound (5.161) for $k = 4$ we get

$$|D^{4,0} J_L(L^{-1} X_{\delta_{n+1}}, 0, 0; g_4)| \le O(1) L^{-(4-\varepsilon)} \|D^{4,0} J(X_{\delta_n}, 0)\| \|g_4\|_{C^2(L^{-1} X^4_{\delta_{n+1}})}$$

which proves the case $(p, m) = (2, 0)$.

Next we turn to the case $(p, m) = (1, 2)$. We have

$$\begin{aligned} & D^{2,2} J_L(L^{-1} X_{\delta_{n+1}}, 0; f^{\times 2}, g_2) \\ &= D^{2,2} J(X_{\delta_n}, 0; f^{\times 2}_{L^{-1}}, g_{2,L^{-1}}) \\ &= D^{2,2} J(X_{\delta_n}, 0; \delta f^{(1)}_{L^{-1}}, f^{(2)}_{L^{-1}}, g_{2,L^{-1}}) + D^{2,2} J(X_{\delta_n}, 0; f^{(1)}_{L^{-1}}(x_0), \delta f^{(2)}_{L^{-1}}, g_{2,L^{-1}}) \\ &\quad + D^{2,2} J(X_{\delta_n}, 0; f^{(1)}_{L^{-1}}(x_0), f^{(2)}_{L^{-1}}(x_0), \delta g_{2,L^{-1}}) \end{aligned} \tag{5.166}$$

where we have used the fourth condition in (4.59). Therefore

$$
\begin{aligned}
&|D^{2,2}J_L(L^{-1}X_{\delta_{n+1}},0,0;\,f^{\times 2},g_2)| \\
&\quad\le \|D^{2,2}J(X_{\delta_n},0,0)\|\Big(\|\delta f^{(1)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|f^{(2)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|g_{2,L^{-1}}\|_{C^2(X^2_{\delta_n})} \\
&\qquad+\|\delta f^{(2)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|f^{(1)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|g_{2,L^{-1}}\|_{C^2(X^2_{\delta_n})} \\
&\qquad+\|\delta g_{2,L^{-1}}\|_{C^2(X^2_{\delta_n})}\|f^{(1)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\|f^{(2)}_{L^{-1}}\|_{C^2(X_{\delta_n})}\Big)
\end{aligned}
$$

Then using the bounds (5.158), (5.161) for $k=1,2$ and (5.162) for $k=1$ we get

$$
\begin{aligned}
&|D^{2,2}J_L(L^{-1}X_{\delta_{n+1}},0,0;\,f^{\times 2},g_2)| \\
&\quad\le O(1)L^{-(4-\varepsilon)}\|D^{2,2}J(X_{\delta_n},0)\|\prod_{j=1}^{2}\|f^{(j)}\|_{C^2(L^{-1}X_{\delta_n})}\|g_2\|_{C^2(L^{-1}X^2_{\delta_{n+1}})}
\end{aligned}
$$

which proves the case $(p,m)=(1,2)$.

Finally we treat the case $(n,m)=(0,4)$. Let $\mathcal{N}_4=(1,2,3,4)$. Then using the fourth condition of (4.59) we get

$$
\begin{aligned}
D^{0,4}J_L(L^{-1}X_{\delta_{n+1}},0,0;\,f^{\times 4}) &= D^{0,4}J(X_{\delta_n},0;\,f^{\times 4}_{L^{-1}}) \\
&= \sum_{I\subset\mathcal{N}_4,|I|\ne 4} D^{0,4}J(X_{\delta_n},0;\,f_{L^{-1}}(x_0)^{\times|I|},\delta f^{\times|I_c|}_{L^{-1}})
\end{aligned}
$$

where $f^{\times|I|}=(f_j)|_{j\in I}$.

Therefore

$$
\begin{aligned}
&|D^{0,4}J_L(L^{-1}X_{\delta_n},0,0;\,f^{\times 4})| \\
&\quad\le \|D^{0,4}J(X_{\delta_n},0,0)\|\sum_{I\subset\mathcal{N}_4,|I|\ne 4}\|f_{L^{-1}}\|^{|I|}_{C^2(X_{\delta_n})}\|\delta f_{L^{-1}}\|^{|I_c|}_{C^2(X_{\delta_n})}
\end{aligned}
$$

Because of the condition on I in the above sum $|I_c|\ge 1$. Therefore using the bounds (5.158) and (5.161) we get

$$
|D^{0,4}J_L(L^{-1}X_{\delta_n},0,0;\,f^{\times 4})|\le O(1)L^{-(4-\varepsilon)}\|D^{0,4}J(X_{\delta_n},0)\|\prod_{j=1}^{4}\|f_j\|_{C^2(L^{-1}X_{\delta_{n+1}})}
$$

which proves the case $(p,m)=(0,4)$ and thus completes the proof of Lemma 5.24. □

Corollary 5.1 *Let $Y_{\delta_{n+1}}=L^{-1}X_{\delta_{n+1}}$ where X is a small set, $Z_{\delta_{n+1}}=L^{-1}\bar{X}^L_{\delta_{n+1}}$ and let $J(X_{\delta_n},\Phi)$ be normalized as in (4.59). By definition $J_L(Y_{\delta_{n+1}},\Phi)=J(X_{\delta_n},S_L\Phi)$. Then*

$$
|J_L(Y_{\delta_{n+1}})|_{\mathbf{h}}\le O(1)L^{-(7-\varepsilon)/2}|J(X_{\delta_n})|_{\hat{\mathbf{h}}} \tag{5.167}
$$

$$
\|J_L(Y_{\delta_{n+1}})e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}})}\|_{\mathbf{h},G_\kappa}\le O(1)L^{-(7-\varepsilon)/2}
$$

$$\times\left[|J(X_{\delta_n})|_{\hat{\mathbf{h}}}+\|J(X_{\delta_n})\|_{\hat{\mathbf{h}},G_{3\kappa}}\right] \tag{5.168}$$

$$|J_L(Y_{\delta_{n+1}})|_{\mathbf{h}_*}\leq O(1)L^{-(7-\varepsilon)/2}|J(X_{\delta_n})|_{\hat{\mathbf{h}}*} \tag{5.169}$$

where (*see Lemma* 5.15) $\hat{\mathbf{h}}=(\hat{h}_F,h_B)$, $\hat{\mathbf{h}}_*=(\hat{h}_F,h_{B*})$ *and* $\hat{h}_F=h_F/2$.

Proof Equations (5.167), (5.169) follow easily from Lemma 5.24 taking advantage of the scaling present to shift from $\mathbf{h}$ to $\hat{\mathbf{h}}$. To see this observe that since $h_F=2\hat{h}_F$ we have from the definition of the $\mathbf{h}$ norm

$$|J_L(Y_{\delta_{n+1}})|_{\mathbf{h}}=\sum_{n=0}^{\infty}\sum_{m=0}^{m_0}\hat{h}_F^{2n}\frac{h_B^m}{m!}2^{2n}\|D^{2n,m}J_L(Y_{\delta_{n+1}},0)\| \tag{5.170}$$

Only terms with $2n+m$ even contribute. For $2n+m\leq 4$ use Lemma 5.24 and observe that $2^{2n}\leq O(1)$. For $2n+m\geq 6$, we have for L sufficiently large and ε sufficiently small depending on L

$$\begin{aligned}2^{2n}\|D^{2n,m}J_L(Y_{\delta_{n+1}},0)\| &\leq 2^{2n}L^{-(2n+m)\frac{(3-\varepsilon)}{4}}\|D^{2n,m}J(X_{\delta_n},0)\|\\ &\leq L^{-(2n+m)\frac{(3-\frac{1}{3}-\varepsilon)}{4}}\|D^{2n,m}J(X_{\delta_n},0)\|\\ &\leq O(1)L^{-(7-\varepsilon)/2}\|D^{2n,m}J(X_{\delta_n},0)\|\end{aligned}$$

Putting the two case together in (5.170) gives (5.167). The proof of (5.169) is the same on replacing h_B by h_{B*}.

For (5.168) we write

$$\begin{aligned}&\|J_L(Y_{\delta_{n+1}},\Phi)e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}},\Phi)}\|_{\mathbf{h}}\\ &\quad\leq\|J_L(Y_{\delta_{n+1}},\Phi)\|_{\mathbf{h}}\|e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}},\Phi)}\|_{\mathbf{h}}\\ &\quad\leq O(1)G_\kappa(Z_{\delta_{n+1}},\phi)\left[|J_L(Y_{\delta_{n+1}})|_{\mathbf{h}}+L^{-m_0d_s}\|J_L(Y_{\delta_{n+1}})\|_{h_F,L^{[\phi]}h_B,G_\kappa}\right]\end{aligned}$$

where we used Lemmas 5.5 and 5.15. By (5.167), and rewriting the second term by moving the scaling from J to the norm,

$$\begin{aligned}&\|J_L(Y_{\delta_{n+1}},\Phi)e^{-\tilde{V}_L(Z_{\delta_{n+1}}\setminus Y_{\delta_{n+1}},\Phi)}\|_{\mathbf{h}}\\ &\quad\leq O(1)G_\kappa(Z_{\delta_{n+1}},\varphi)\left[L^{-(7-\varepsilon)/2}|J(X_{\delta_n})|_{\hat{\mathbf{h}}}+L^{-m_0d_s}\|J(X_{\delta_n})\|_{\hat{\mathbf{h}},G_{3\kappa}}\right]\end{aligned}$$

Recall that $m_0=9$ and the scaling dimension $d_s=(3-\varepsilon)/4$. Equation (5.168) now follows by multiplying both sides by $G_\kappa^{-1}(Z_{\delta_{n+1}},\varphi)$, and taking the supremum over φ. □

Lemma 5.25

$$\|\tilde{F}_{R_n}e^{-\tilde{V}_n}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_n}\leq O(1)\varepsilon^{3/4-\eta} \tag{5.171}$$

$$|\tilde{F}_{R_n}e^{-\tilde{V}_n}|_{\hat{\mathbf{h}_*},\mathcal{A},\delta_n}\leq O(1)\varepsilon^{11/4-\eta} \tag{5.172}$$

and $J_n = R_n^\sharp - \tilde{F}_{R_n} e^{-\tilde{V}_n}$ *satisfies on small sets the bounds*

$$\begin{aligned} \|J_n\|_{\hat{\mathbf{h}},G_{3\kappa},\mathcal{A},\delta_n} &\le O(1)\bar{g}^{\frac{3}{4}-\eta} \\ |J_n|_{\hat{\mathbf{h}},\mathcal{A},\delta_n} &\le O(1)\bar{g}^{\frac{3}{4}-\eta} \\ |J_n|_{\hat{\mathbf{h}}_*,\mathcal{A},\delta_n} &\le O(1)\bar{g}^{\frac{11}{4}-\eta} \end{aligned} \tag{5.173}$$

Proof First we prove (5.171). $\tilde{F}_{R_n}$ is defined in (4.43) and (4.45), and is supported on small sets. We estimate its $\mathbf{h}$ norm as in the proof of Lemma 5.19

$$\begin{aligned} \|\tilde{F}_{R_n}(X_{\delta_n},\Phi)\|_{\mathbf{h}} &\le \sum_P |\tilde{\alpha}_{n,P}(X_{\delta_n})| \int_{X_{\delta_n}} dx\, \|P(\Phi(x),\partial\Phi(x))\|_{\mathbf{h}} \\ &\le C_L \sum_P |\tilde{\alpha}_{n,P}(X_{\delta_n})|\bar{g}^{-1}\left(\bar{g}\int_{X_{\delta_n}} dx\, (|\varphi|^2)^2(x) + \bar{g}^{1/2}\|\varphi\|^2_{X_{\delta_n},1,\sigma} + 1\right) \\ &\le C_L \sum_P |\tilde{\alpha}_{n,P}(X_{\delta_n})|\bar{g}^{-1} G_\kappa(X_{\delta_n},\varphi) e^{\gamma\bar{g}\int_{X_{\delta_n}} dy (|\varphi|^2)^2(y)} \end{aligned}$$

for any $\gamma = O(1) > 0$. Hence, using Lemma 5.5

$$\begin{aligned} \|\tilde{F}_{R_n}(X_{\delta_n},\Phi)e^{-\tilde{V}_n(X_{\delta_n},\Phi)}\|_{\mathbf{h}} &\le \|\tilde{F}_{R_n}(X_{\delta_n},\Phi)\|_{\mathbf{h}} \|e^{-\tilde{V}_n(X_{\delta_n},\Phi)}\|_{\mathbf{h}} \\ &\le C_L \sum_P 2^{|X_{\delta_n}|}|\tilde{\alpha}_{n,P}(X_{\delta_n})|\bar{g}^{-1} G_\kappa(X_{\delta_n},\varphi) \end{aligned}$$

We thus obtain (remembering that $\tilde{\alpha}_{n,P}$ are supported on small sets) on using (5.136) of Lemma 5.17 for ε sufficiently small depending on L, implying $\bar{g}$ sufficiently small,

$$\|\tilde{F}_{R_n}(X_{\delta_n},\Phi)e^{-\tilde{V}_n(X_{\delta_n},\Phi)}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_n} \le C_L\bar{g}^{-1}\sum_P \|\tilde{\alpha}_{n,P}\|_{\mathcal{A},\delta_n} \le O(1)\bar{g}^{3/4-\eta}$$

This proves (5.171). Now we turn to the proof of (5.172). As observed in the proof of Lemma 5.17, $\hat{\mathbf{h}}_*^{n_P}|\tilde{\alpha}_{n,P}|_{\mathcal{A},\delta_n} \le n(P)!\,|1_S R_n^\sharp|_{\hat{\mathbf{h}}_*,\mathcal{A},\delta_n} \le O(1)\bar{g}^{11/4-\eta}$. We have from the definition of $\tilde{F}_{R_n}$ given in (4.43) $|\tilde{F}_{R_n}(X_{\delta_n})|_{\hat{\mathbf{h}}_*} \le O(1)\sum_P |\tilde{\alpha}_{n,P}(X_{\delta_n})|\hat{\mathbf{h}}_*^{n_P}$, whence

$$|\tilde{F}_{R_n}|_{\hat{\mathbf{h}}_*,\mathcal{A},\delta_n} \le \sum_P |\tilde{\alpha}_{n,P}|_{\mathcal{A},\delta_n}\hat{\mathbf{h}}_*^{n_P} \le O(1)\bar{g}^{11/4-\eta}$$

which proves (5.172).

To get these bounds for $J_n = R_n^\sharp - \tilde{F}_{R_n}e^{-\tilde{V}_n}$ we use (5.171) and (5.172) to bound $\tilde{F}_{R_n}e^{-\tilde{V}_n}$ part. We can substitute $\hat{\mathbf{h}}$ for $\mathbf{h}$ in (5.171) since the $\hat{\mathbf{h}}$ norm is smaller than the $\mathbf{h}$ norm. We bound $R_n^\sharp$ by Lemma 5.17. We have also used the trivial bound $|J|_{\hat{\mathbf{h}},\mathcal{A},\delta_n} \le \|J\|_{\hat{\mathbf{h}},G_{3\kappa},\mathcal{A},\delta_n}$ to obtain the second inequality for J from the first. □

Lemma 5.26

$$\|R_{n+1,\text{linear}}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \le O(1)L^{-(1-\varepsilon)/2}\bar{g}^{3/4-\eta} \tag{5.174}$$

$$|R_{n+1,\text{linear}}|_{\mathbf{h}_*,\mathcal{A},\delta_{n+1}} \le O(1)L^{-(1-\varepsilon)/2}\bar{g}^{11/4-\eta} \tag{5.175}$$

Remark This is a crucial lemma which also figures as Lemma 5.27 in [13] and its proof is the same. For the readers benefit we give the details below. The proof is based on the principle that the contribution to the linearized part of the remainder from large sets is very small. For small sets the expanding contributions have been subtracted out leading to normalized polymer activities and this is sufficient to provide a contracting factor.

Proof Let $R_{n+1,\mathrm{linear}}$, given in (4.47) is the sum of two terms which represent contributions from small/large sets respectively. Let $R_{n+1,\mathrm{linear,s.s}}$ denote the first term:

$$R_{n+1,\mathrm{linear,s.s}}(Z_{\delta_{n+1}}) = \sum_{\substack{X_{\delta_{n+1}}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}} e^{-\tilde{V}_L(Z_{\delta_{n+1}}\backslash Y_{\delta_{n+1}})} J_{n,L}(Y_{\delta_{n+1}}) \tag{5.176}$$

where $Y_{\delta_{n+1}} = L^{-1}X_{\delta_{n+1}}$ and $J_n = R^\sharp_n - \tilde{F}_{R_n}e^{-\tilde{V}_n}$. By Corollary 5.1 we get

$$\begin{aligned}&\|R_{n+1,\mathrm{linear,s.s}}(Z_{\delta_{n+1}})\|_{\mathbf{h},G_\kappa}\\ &\quad\le O(1)L^{-(7-\varepsilon)/2}\sum_{\substack{X_{\delta_{n+1}}:\text{small sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}}\Big[|J_n(X_{\delta_n})|_{\hat{\mathbf{h}}} + \|J(X_{\delta_n})\|_{\hat{\mathbf{h}},G_{3\kappa}}\Big]\end{aligned} \tag{5.177}$$

Note that $Z_{\delta_{n+1}}$ fixes Z_{δ_n} by restriction and the sum on the right hand side is the same as the sum over X_{δ_n} such that $L^{-1}\bar{X}^L_{\delta_n} = Z_{\delta_n}$. We multiply both sides by $\mathcal{A}(Z_{\delta_{n+1}})$. On the right hand side we have $\mathcal{A}(Z_{\delta_{n+1}}) = \mathcal{A}(Z_{\delta_n}) = \mathcal{A}(\bar{X}^L_{\delta_n}) \le O(1)\mathcal{A}(X_{\delta_n})$ by (2.10). We fix a unit block Δ_{n+1} and sum over $Z_{\delta_{n+1}} \supset \Delta_{n+1}$. This fixes by restriction to the over $Z_{\delta_n} \supset \Delta_n$ on the right hand side. The argument on p. 790 of [6] controls the constrained sum on X_{δ_n} such that $L^{-1}\bar{X}^L_{\delta_n} = Z_{\delta_n} \supset \Delta_n$ by L^3 times the sum over $X_{\delta_n} \supset \Delta_n$. Taking then the supremum over the fixed unit block gives

$$\begin{aligned}\|R_{n+1,\mathrm{linear,s.s}}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} &\le O(1)L^{-(7-\varepsilon)/2}L^3\big[|J_n|_{\hat{\mathbf{h}},\mathcal{A},\delta_n} + \|J_n\|_{\hat{\mathbf{h}},G_{3\kappa},\mathcal{A},\delta_n}\big]\\ &\le O(1)L^{-(1-\varepsilon)/2}\bar{g}^{3/4-\eta}\end{aligned} \tag{5.178}$$

where for the second inequality we used Lemma 5.25.

The second term in (4.47) for $R_{n+1,\mathrm{linear}}$ which gets contributions only from large sets is

$$R_{n+1,\mathrm{linear,l.s.}}(Z_{\delta_{n+1}}) = \sum_{\substack{X_{\delta_{n+1}}:\text{large sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}} e^{-\tilde{V}_L(Z_{\delta_{n+1}}\backslash Y_{\delta_{n+1}})} R^\natural_{n,L}(L^{-1}X_{\delta_{n+1}}) \tag{5.179}$$

where we have used $J_n = R^\sharp_n$ since the relevant part $\tilde{F}_{R_n}$ is supported on small sets. We first bound in the $\mathbf{h}$ norm and observe that because of the rescaling involved and Lemma 5.17

$$\|R^\natural_{n,L}(L^{-1}X_{\delta_{n+1}})\|_{\mathbf{h}} \le \|R^\sharp_n(X_{\delta_n})\|_{\hat{\mathbf{h}},G_{3\kappa}}G_\kappa(X_{\delta_{n+1}}) \le O(1)2^{|X_{\delta_n}|}\bar{g}^{3/4-\eta}G_\kappa(Z_{\delta_{n+1}})$$

so that on using Lemma 5.5 for $e^{-\tilde{V}_L}$

$$\|R_{n+1,\mathrm{linear,l.s.}}(Z_{\delta_{n+1}})\|_{\mathbf{h},G_\kappa} \le O(1)\bar{g}^{3/4-\eta}\sum_{\substack{X_{\delta_{n+1}}:\text{large sets}\\ L^{-1}\bar{X}^L_{\delta_{n+1}}=Z_{\delta_{n+1}}}} 2^{2|X_{\delta_{n+1}}|}$$

We estimate the $\mathcal{A}$ norm as before except that for large sets we use from (2.11).

$\mathcal{A}(L^{-1}\bar{X}^L_{\delta_{n+1}}) \le c_p L^{-4}\mathcal{A}_{-p}(X_{\delta_{n+1}})$ for any positive integer p with $c_p = O(1)$. Choose $p = 2$. Therefore

$$\|R_{n+1,\mathrm{linear,l.s.}}\|_{\mathbf{h},G_\kappa,\mathcal{A},\delta_{n+1}} \le O(1)L^{-1}\bar{g}^{3/4-\eta} \tag{5.180}$$

Adding the contributions (5.176) and (5.180) we get (5.174). Equation (5.175) can be proved in the same way. For the small set contribution we use the kernel bounds in Corollary 5.1 and Lemma 5.25. For the large set contribution we first use the rescaling involved to shift on the right hand side the $\mathbf{h}_*$ norm to the $\hat{\mathbf{h}}_*$ norm followed by the kernel bound in Lemma 5.17. □

Proof of Theorem 5.1 From (4.41), R_{n+1} is the sum of $R_{n+1,\mathrm{main}}$, $R_{n+1,\mathrm{linear}}$, $R_{n+1,3}$ and $R_{n+1,4}$. $R_{n+1,\mathrm{main}}$ satisfies the bound given in Lemma 5.21. For L large and ε small depending on L implying $\bar{g}$ sufficiently small $C_L\bar{g}^{3/4} \le L^{-1/2}\bar{g}^{3/4-\eta}$ with $\eta = 1/64$. Similarly $C_L\bar{g}^{3-3\delta/2} \le L^{-1/2}\bar{g}^{11/4-\eta}$ for $\delta = \eta$. Therefore $|||R_{n+1,\mathrm{main}}|||_{n+1} \le L^{-1/2}\bar{g}^{11/4-\eta}$. Similarly from Lemmas 5.22 and 5.23 we get $|||R_{n+1,j}|||_{n+1} \le L^{-1/2}\bar{g}^{11/4-\eta}$, for $j = 3, 4$. Adding these bounds to that provided for $R_{n+1,\mathrm{main}}$ by Lemma 5.26 we have that the sum satisfies the bound (5.11) for L sufficiently large. The bounds (5.9), (5.10) and (5.12) have been proved in Lemmas 5.17 and 5.18. □

6 Existence of the Global Renormalization Group Trajectory and the Stable Manifold

This section is devoted to the proof of existence of the stable manifold starting from the unit lattice. Namely, there exists an initial critical mass μ_0 which is a Lipshitz continuous function of the coupling constant g_0 such that RG trajectory is bounded uniformly on all scales. The proof is complicated because of the presence of lattice artifacts which become inocuous if we advance sufficiently on the RG trajectory. We therefore prove the result by a combination of three theorems, namely Theorems 6.1, 6.2, and 6.3. We first iterate the RG map a finite number n_0 of times and then restart the trajectory. Theorem 6.1 says that there exists a critical mass such that the RG trajectory is uniformly bounded at all scales $n \ge n_0 \ge 0$. Theorem 6.2 says that if n_0 is sufficiently large then the critical mass μ_{n_0} at this scale is a Lipshitz continuous function continuous function of the contracting variables. The stable (critical) manifold at scale n_0, appropriately interpreted for a sequence of non-autonomous maps, is constructed. Finally Theorem 6.3 says that there exists an initial critical mass μ_0 which is a C^1 function of g_0 such that after n_0 applications of the RG map we arrive at the critical mass μ_{n_0} of Theorem 6.2. Combining Theorem 6.2 with Theorem 6.3 proves the existence of the stable manifold starting from the unit lattice. One consequence is that the coupling constant g_n is bounded away from 0 uniformly in n.

6.1 The flow

Define $\tilde{g}_n = g_n - \bar{g}$ with $\bar{g}$ defined by (5.2) and $\tilde{\mathbf{w}}_n = \mathbf{w}_n - \mathbf{w}_*$. Here $\mathbf{w}_*$ is the function on $\bigcup_{n\ge0}(\delta_n\mathbb{Z})^3$ defined in Lemma 5.9. According to Lemma 5.9, $\tilde{\mathbf{w}}_n \to 0$ geometrically fast in $\mathcal{W}_l$ for all $l \ge 0$.

We will use as coordinates of the RG trajectory

$$\upsilon_n = (\tilde{g}_n, \mu_n, R_n, \tilde{\mathbf{w}}_n) \tag{6.1}$$

The RG map

$$\upsilon_{n+1} = f_{n+1}(\upsilon_n) \tag{6.2}$$

can be written in components (see (4.39), (4.42), (4.16) and Lemma 5.9)

$$\tilde{g}_{n+1} = f_{n+1,g}(\upsilon_n) = \alpha(\epsilon)\tilde{g}_n + \tilde{\xi}_n(\upsilon_n) \tag{6.3}$$

$$\mu_{n+1} = f_{n+1,\mu}(\upsilon_n) = L^{\frac{3+\varepsilon}{2}}\mu_n + \tilde{\rho}_n(\upsilon_n) \tag{6.4}$$

$$R_{n+1} = f_{n+1,R}(\upsilon_n) =: U_{n+1}(\upsilon_n) \tag{6.5}$$

$$\tilde{\mathbf{w}}_{n+1} = f_{n+1,w}(\upsilon_n) = (\mathbf{v}_{n+1} - \mathbf{v}_{c,*}) + \tilde{\mathbf{w}}_{n,L} \tag{6.6}$$

with initial $\mathbf{w}_0 = 0$ and $R_0 = 0$. This implies that $\tilde{\mathbf{w}}_0 = -\mathbf{w}_*$ and our initial condition is

$$\upsilon_0 = (\tilde{g}_0, \mu_0, 0, \tilde{\mathbf{w}}_0) \tag{6.7}$$

$\alpha(\varepsilon),\ \xi_n,\ \rho_n$ are defined by

$$\alpha(\epsilon) = 2 - L^{\varepsilon} = 1 - O(\log L)\varepsilon$$

for ε sufficiently small depending on L, and

$$\tilde{\xi}_n(\upsilon_n) = -L^{2\varepsilon}a_{c*}\tilde{g}_n^2 - L^{2\varepsilon}(a_n - a_c*)(\tilde{g}_n + \bar{g})^2 + \xi_n(\upsilon_n) \tag{6.8}$$

$$\tilde{\rho}_n(\upsilon_n) = -L^{2\varepsilon}b_n(\bar{g} + \tilde{g}_n)^2 + \rho_n(\upsilon_n) \tag{6.9}$$

Let E_n be the Banach space consisting of elements υ_n with the norm

$$\|\upsilon_n\|_n = \max((\nu\bar{g})^{-1}|\tilde{g}_n|,\ \bar{g}^{-(2-\delta)}|\mu_n|,\ \bar{g}^{-(11/4-\eta)}|||R_n|||_n, \tilde{c}_L^{-1}\|\tilde{\mathbf{w}}_n\|_n) \tag{6.10}$$

where the norm $|||R|||_{\delta_n}$ of R_n is as defined in (5.6). ν is the $O(1)$ constant which figures in the specification of the domain $\mathcal{D}_n$ (5.4). We have $0 < \nu < 1$. We will take $\nu > 0$ sufficiently small depending on L and this will be specified in the proofs of Lemmas 6.2 and 6.3 below. The norm $\|\tilde{\mathbf{w}}_n\|_{\delta_n}$ and the constant $\tilde{c}_L$ are as specified in Lemma 5.9.

6.2 Domains and Bounds

Let $E_n(r) \subset E_n$ be the open ball of radius r, centered at the origin:

$$E_n(r) = \{\upsilon_n \in E_n : \|\upsilon_n\|_n < r\} \tag{6.11}$$

Let $\mathcal{D}_n$ be the domain of (g_n, μ_n, R_n) defined in (5.4) and (5.5).

$$\upsilon_n \in E_n(1) \quad\Longrightarrow\quad (g_n, \mu_n, R_n) \in \mathcal{D}_n \tag{6.12}$$

and then Theorem 5.1 holds.

Let $\upsilon_n \in E_n(1)$. Let $\varepsilon > 0$ be sufficiently small (depending on L). Then from Theorem 5.1, Lemma 5.12, (6.8) and (6.9) we get the bounds

$$\begin{aligned} |\tilde{\xi}_n(\upsilon_n)| &\le C_L\Big((\nu^2 + L^{-nq})\bar{g}^2 + \bar{g}^{11/4-\eta}\Big) \\ |\tilde{\rho}_n(\upsilon_n)| &\le C_L\bar{g}^2 \\ |||U_{n+1}(\upsilon_n)|||_{n+1} &\le L^{-1/4}\bar{g}^{11/4-\eta} \end{aligned} \tag{6.13}$$

We have the following Lipshitz bounds:

Lemma 6.1 *Let $\upsilon_n, \upsilon'_n \in E_n(1/4)$. Then we have*:

$$\begin{aligned}
&\text{(i)} \quad |\tilde{\xi}_n(\upsilon_n) - \tilde{\xi}_n(\upsilon'_n)| \le C_L\Big((\nu^2 + L^{-nq})\bar{g}^2 + \bar{g}^{11/4-\eta}\Big)\|\upsilon_n - \upsilon'_n\|_n \\
&\text{(ii)} \quad |\tilde{\rho}_n(\upsilon_n) - \tilde{\rho}_n(\upsilon'_n)| \le C_L\bar{g}^2\|\upsilon_n - \upsilon'_n\|_n \\
&\text{(iii)} \quad |||U_{n+1}(\upsilon_n) - U_{n+1}(\upsilon'_n)|||_{n+1} \le O(1)L^{-1/4}\bar{g}^{11/4-\eta}\|\upsilon_n - \upsilon'_n\|_n \\
&\text{(iv)} \quad \tilde{c}_L^{-1}\|f_{n+1,\mathbf{w}}(\upsilon_n) - f_{n+1,\mathbf{w}}(\upsilon'_n)\|_{n+1} \le L^{-1/5}\|\upsilon_n - \upsilon'_n\|_n
\end{aligned} \tag{6.14}$$

Proof $\tilde{\xi}_n, \tilde{\rho}_n, U_{n+1}$ are (norm) analytic functions in $\mathcal{D}_n$ and thus in $E_n(1)$. The analyticity follows from the algebraic operations in Sect. 4, the norm analyticity of the reblocking map together with the norm analyticity of the extraction map (Theorem 5 [6]). Therefore we can use Cauchy estimates exactly as in the proof of Lemma 6.1 of [13] together with the bounds (6.13) to get (i), (ii) and (iii). To get (iv) note that from (6.6) and the definition of the norms in (5.61) we have

$$\begin{aligned}
&\|f_{n+1,\mathbf{w}}(\upsilon_n) - f_{n+1,\mathbf{w}}(\upsilon'_n)\|_{n+1} \\
&\quad \le L^{2d_s} \max_{1\le p\le 3} \sup_{x\in(\delta_{n+1}\mathbf{Z})^3} \Big((|x| + \delta_{n+1})^{\frac{6p+1}{4}} |\tilde{w}_n^{(p)}(Lx) - \tilde{w}_n'^{(p)}(Lx)|\Big) \\
&\quad \le L^{2d_s} \max_{1\le p\le 3} \sup_{y\in(\delta_n\mathbf{Z})^3} \left(\left(\frac{|y|}{L} + \delta_{n+1}\right)^{\frac{6p+1}{4}} |\tilde{w}_n^{(p)}(y) - \tilde{w}_n'^{(p)}(y)|\right) \\
&\quad \le L^{2d_s - 7/4} \max_{1\le p\le 3} \sup_{y\in(\delta_n\mathbf{Z})^3} \Big((|y| + \delta_n)^{\frac{6p+1}{4}} |\tilde{w}_n^{(p)}(y) - \tilde{w}_n'^{(p)}(y)|\Big) \\
&\quad \le L^{-1/5}\|\tilde{\mathbf{w}}_n - \tilde{\mathbf{w}}'_n\|_n
\end{aligned}$$

where we have used $d_s = \frac{3-\varepsilon}{4}$ and then L large and ε sufficiently small. Now dividing both sides by $\tilde{c}_L$ we get (iv). □

6.3 Existence of the Global RG Trajectory

Let $\mathcal{D}_n$ be the domain specified by (5.4), (5.5), and (5.6). Let $(\tilde{g}_0, \mu_0, 0)$ belong to $\tilde{\mathcal{D}}_0 \subset \mathcal{D}_0$ where $\tilde{\mathcal{D}}_0$ is specified by

$$|\tilde{g}_0| < 2^{-(n_0+5)}\nu\bar{g}, \qquad |\mu_0| < 2^{-(n_0+5)}L^{-\frac{3+\varepsilon}{2}n_0}\bar{g}^{2-\delta}$$

Let n_0 be a positive integer. By iterating the RG map n_0 times using Theorem 5.1 and the flow equation (4.39) recursively we obtain for ε sufficiently small depending on L and n_0, $(g_{n_0}, \mu_{n_0}, R_{n_0}) \in \mathcal{D}_{n_0}(1/32)$ where $\mathcal{D}_{n_0}(1/32)$ is specified by

$$|\tilde{g}_{n_0}| < \frac{1}{32}\nu\bar{g}, \qquad |\mu_{n_0}| < \frac{1}{32}\bar{g}^{2-\delta}$$
$$|||R_{n_0}|||_{n_0} < \frac{1}{32}\bar{g}^{11/4-\eta}$$

We will now prove the existence of a global solution to the discrete flow map (6.2):

$$\upsilon_{n+1} = f_{n+1}(\upsilon_n), \quad \forall\, n \ge n_0$$

with initial condition

$$\upsilon_{n_0} = (\tilde{g}_{n_0}, \mu_{n_0}, \tilde{\mathbf{w}}_{n_0}, R_{n_0})$$

in a bounded domain. We will say that $\{\upsilon_n : \upsilon_{n+1} = f_{n+1}(\upsilon_n),\ n \ge n_0\}$ is the RG trajectory *restarted* at scale n_0. To this end we consider the Banach space $\mathbf{E}_{n_0}$ of sequences $\mathbf{s}_{n_0} = \{\upsilon_n\}_{n\ge n_0}$, each $\upsilon_n \in E_n$, with the norm

$$\|\mathbf{s}_{n_0}\| = \sup_{n\ge n_0} \|\upsilon_n\|_n \tag{6.15}$$

and the open ball $\mathbf{E}_{n_0}(r) \subset \mathbf{E}$

$$\mathbf{E}_{n_0}(r) = \{\mathbf{s}_{n_0} : \|\mathbf{s}_{n_0}\| < r\} \tag{6.16}$$

We will derive on the space of sequences $\mathbf{E}_{n_0}$ an equation that a global RG trajectory must solve and then prove for $\upsilon_{n_0} \in E_{n_0}(1/32)$ the existence of a unique solution in the ball $\mathbf{E}_{n_0}(1/4)$, for a suitable choice of r, by the contraction mapping principle. This adapts a standard method from the theory of hyperbolic dynamical systems in Banach spaces due to Irwin in [23]. Irwin's analysis is explained by Shub in Appendix 2, Chap. 5 of [27]. For earlier applications see Sect. 5 of [7] and Sect. 6 of [13].

Theorem 6.1 *Let L be large, ν be sufficiently small depending on L, then ε sufficiently small depending on L. Let $\upsilon_{n_0} \in E_{n_0}(1/32)$ for any integer $n_0 \ge 0$. Let $(\tilde{g}_{n_0}, R_{n_0}, \tilde{\mathbf{w}}_{n_0})$ be held fixed. Then there is a μ_{n_0} such that there exists a sequence $\mathbf{s}_{n_0} = \{\upsilon_n\}_{n\ge n_0}$ in $\mathbf{E}_{n_0}(1/4)$ satisfying $\upsilon_{n+1} = f_{n+1}(\upsilon_n)$ for all $n \ge n_0$.*

Remark The μ_{n_0} of the theorem is called a *critical mass*. We write $\mu_{n_0} = \mu_{n_0,c}$.

Proof Our initial data will be at scale n_0. Let $n_0 \le n \le N-1$. We iterate the map (6.3) forwards N times. We iterate the map (6.4) backwards $N - n_0$ times starting from a given μ_N. We then easily derive

$$\tilde{g}_{n+1} = \alpha(\epsilon)^{n+1-n_0}\tilde{g}_{n_0} + \sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j}\tilde{\xi}_j(\upsilon_j), \quad n_0 \le n \le N-1$$

$$\mu_{n+1} = L^{-\frac{3+\varepsilon}{2}(N-n-1)}\mu_N - \sum_{j=n+1}^{N-1} L^{-\frac{3+\varepsilon}{2}(j-n)}\tilde{\rho}_j(\upsilon_j), \quad n_0 - 1 \le n \le N-2$$

Let us fix $\mu_N = \mu_f$ and take $N \to \infty$. In other words we assume the μ_n flow is bounded and then must show that such a flow exists. We have

$$\tilde{g}_{n+1} = \alpha(\epsilon)^{n+1-n_0}\tilde{g}_{n_0} + \sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j}\tilde{\xi}_j(\upsilon_j), \quad n \ge n_0 \tag{6.17}$$

$$\mu_{n+1} = -\sum_{j=n+1}^{\infty} L^{-\frac{3+\varepsilon}{2}(j-n)}\tilde{\rho}_j(\upsilon_j), \quad n \ge n_0 - 1 \tag{6.18}$$

together with

$$R_{n+1} = U_{n+1}(\upsilon_n), \quad n \ge n_0 \tag{6.19}$$

$$\tilde{\mathbf{w}}_{n+1} = (\mathbf{v}_{n+1} - \mathbf{v}_{c*}) + \tilde{\mathbf{w}}_{n,L} \tag{6.20}$$

The flow $\tilde{\mathbf{w}}_n$ is independent of that of g_n, μ_n, R_n, is solved by (5.60) and satisfies the bounds of Lemma 5.9. This solution can be incorporated in the υ_n and $\tilde{\mathbf{w}}_n$ is then no longer a flow variable.

For ε sufficiently small (depending on L)

$$0 < \alpha(\epsilon) < 1 \tag{6.21}$$

Note that $\mathbf{s}_{n_0} \in \mathbf{E}_{n_0}(1/4)$ implies $\upsilon_j \in E_j(1/4)$ for all $j \geq n_0$. Then the infinite sum of (6.18) converges by (6.21) and (6.13). So μ_{n_0} has now been determined provided (6.17)–(6.19) has a solution in the afore mentioned ball. It is easy to verify that any solution of (6.17)–(6.18), together with the $\tilde{\mathbf{w}}$ flow, is a solution of the RG flow $\upsilon_{n+1} = f_{n+1}(\upsilon_n)$ for $n \geq n_0$.

We write (6.17)–(6.19) in the form

$$\upsilon_{n+1} = F_{n+1}(\mathbf{s}_{n_0}), \quad n \geq n_0 \tag{6.22}$$

where $\mathbf{s}_{n_0} = (\upsilon_{n_0}, \upsilon_{n_0+1}, \upsilon_{n_0+2}, \ldots)$ and F_{n+1} has components $(F^{(g)}_{n+1}, F^{(\mu)}_{n+1}, F^{(R)}_{n+1})$ given by the r.h.s. of (6.17), (6.18), (6.19) respectively.

If we write

$$\mathbf{F}(\mathbf{s}_{n_0}) = (F_{n_0}(\mathbf{s}_{n_0}), F_{n_0+1}(\mathbf{s}_{n_0}), \ldots)$$

then (6.22) can be written as a fixed point equation

$$\mathbf{s}_{n_0} = \mathbf{F}(\mathbf{s}_{n_0}) \tag{6.23}$$

We seek a solution of (6.23) in the open ball $\mathbf{E}_{n_0}(1/4)$ with initial data $\upsilon_{n_0} = (\tilde{g}_{n_0}, \mu_{n_0}, R_{n_0}, \tilde{\mathbf{w}}_{n_0})$ in $E_{n_0}(1/32)$ with $(\tilde{g}_{n_0}, R_{n_0}, \tilde{\mathbf{w}}_{n_0})$ held fixed. The existence of a unique solution follows by the standard contraction mapping principle and the next lemma. □

Lemma 6.2

$$\mathbf{s}_{n_0} \in \mathbf{E}_{n_0}(1/32) \quad \Longrightarrow \quad \mathbf{F}(\mathbf{s}_{n_0}) \in \mathbf{E}_{n_0}(1/16) \tag{6.24}$$

Moreover, for $\mathbf{s}_{n_0}, \mathbf{s}'_{n_0} \in \mathbf{E}_{n_0}(1/4)$

$$\|\mathbf{F}(\mathbf{s}_{n_0}) - \mathbf{F}(\mathbf{s}'_{n_0})\| \leq \frac{1}{2}\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\| \tag{6.25}$$

Proof First we prove (6.24), and thus take $\mathbf{s}_{n_0} \in \mathbf{E}_{n_0}(1/32)$. Then $\upsilon_n \in E_n(1/32)$ for every $n \geq n_0$ and we can use the estimates (6.13). From (6.17) and the estimates in (6.13) we have

$$\begin{aligned}
(\nu\bar{g})^{-1}|F^{(g)}_{n+1}(\mathbf{s}_{n_0})| &< \alpha(\epsilon)\frac{1}{32} + (\nu\bar{g})^{-1}\sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j} C_L\Big((\nu^2 + L^{-jq})\bar{g}^2 + \bar{g}^{11/4-\eta}\Big) \\
&< \frac{1}{32} + C_L\frac{\bar{g}^{7/4-\eta}}{\nu(1-\alpha(\epsilon)}) + C_L\frac{\bar{g}\nu}{1-\alpha(\epsilon)} + C_L\frac{\bar{g}}{\nu(1-\alpha(\varepsilon)^{-1}L^{-q})} \\
&< \frac{1}{32} + \frac{C_L}{\nu\log L}\varepsilon^{3/4-\eta} + \frac{C_L}{\log L}\nu + \frac{C_L}{\nu(1-L^{-q})}\varepsilon < \frac{1}{16}
\end{aligned}$$

for L sufficiently large, ν sufficiently small depending on L so that $\frac{C_L}{\log L}\nu \le 1/96$ and then ε sufficiently small depending on L so that $\bar{g} \le C_L\varepsilon$ is sufficiently small, $\frac{C_L}{\nu(1-L^{-q})}\varepsilon \le 1/96$ and $\frac{C_L}{\nu \log L}\varepsilon^{1/4-\eta} \le 1/96$.

Similarly from (6.18) and (6.13) we have

$$\bar{g}^{-(2-\delta)}|F_{n+1}^{(\mu)}(\mathbf{s}_{n_0})| \le c_L\bar{g}^{\delta}\sum_{j=n+1}^{\infty} L^{-\frac{3+\varepsilon}{2}(j-n)} \le L^{-\frac{3+\varepsilon}{2}}(1-L^{-\frac{3+\varepsilon}{2}})^{-1} < \frac{1}{16}$$

since $\delta = 1/64$, L sufficiently large, and ε sufficiently small depending on L.

Finally from (6.19) and (6.13)

$$\bar{g}^{-(11/4-\eta)}|||F_{n+1}^{(R)}(\mathbf{s}_{n_0})|||_{n+1} \le L^{-1/4} < \frac{1}{16}$$

for L sufficiently large. This proves (6.24).

To prove (6.25), take $\mathbf{s}_{n_0}, \mathbf{s}'_{n_0} \in \mathbf{E}_{n_0}(1/4)$. This implies that $\upsilon_n, \upsilon'_n \in E_n(1/4)$ for every $n \ge n_0$ and we can use the Lipshitz estimates of Lemma 6.1. Note that the initial coupling g_{n_0} is held fixed. Then we have

$$\begin{aligned}
&(\nu\bar{g})^{-1}|F_{n+1}^{(g)}(\mathbf{s}_{n_0}) - F_{n+1}^{(g)}(\mathbf{s}'_{n_0})| \\
&\quad \le \sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j}(\nu\bar{g})^{-1}|\tilde{\xi}_j(\upsilon_j) - \tilde{\xi}_j(\upsilon'_j)| \\
&\quad \le (\nu\bar{g})^{-1}\sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j} C_L\Big((\nu^2 + L^{-jq})\bar{g}^2 + \bar{g}^{11/4-\eta}\Big)\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\| \\
&\quad \le \left(\frac{C_L}{\nu\log L}\varepsilon^{1/4-\eta} + C_L\frac{\bar{g}\nu}{1-\alpha(\epsilon)} + C_L\frac{\bar{g}}{\nu(1-\alpha(\varepsilon)^{-1}L^{-q})}\right)\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\| \le \frac{1}{2}\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\|
\end{aligned}$$

by estimating as above in the bound for $F_{n+1}^{(g)}(\mathbf{s}_{n_0})$ with L sufficiently large, ν sufficiently small depending on L and ε sufficiently small depending on L. Similarly,

$$\begin{aligned}
&\bar{g}^{-(2-\delta)}|F_{n+1}^{(\mu)}(\mathbf{s}_{n_0}) - F_{n+1}^{(\mu)}(\mathbf{s}'_{n_0})| \\
&\quad \le \sum_{j=n+1}^{\infty} L^{-\frac{3+\varepsilon}{2}(j-n)}\bar{g}^{-(2-\delta)}|\tilde{\rho}_j(\upsilon_j) - \tilde{\rho}_j(\upsilon'_j)| \\
&\quad \le L^{-\frac{3+\varepsilon}{2}}(1-L^{-\frac{3+\varepsilon}{2}})^{-1}C_L\bar{g}^{\delta}\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\| \le \frac{1}{2}\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\|
\end{aligned}$$

for L sufficiently large and ε sufficiently small depending on L. Finally

$$\begin{aligned}
&\bar{g}^{-(11/4-\eta)}|||F_{n+1}^{(R)}(\mathbf{s}_{n_0}) - F_{n+1}^{(R)}(\mathbf{s}'_{n_0})|||_{n+1} \\
&\quad = \bar{g}^{-(11/4-\eta)}|||U_{n+1}(\upsilon_n) - U_{n+1}(\upsilon'_n)|||_{n+1} \\
&\quad \le O(1)L^{-1/4}\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\| \le \frac{1}{2}\|\mathbf{s}_{n_0} - \mathbf{s}'_{n_0}\|
\end{aligned}$$

for L sufficiently large. Thus (6.25) has been proved. This completes the proof of Theorem 6.1. □

6.4 Stable manifold

Theorem 6.1 says that if $\upsilon_{n_0} = (\tilde{g}_{n_0}, \mu_{n_0}, R_{n_0}, \tilde{\mathbf{w}}_{n_0}) \in E_{n_0}(1/32)$ for any $n_0 \geq 0$ then there is a critical mass $\mu_{n_0} = \mu_{n_0,c}$ such that a uniformly bounded RG trajectory exists. The Theorem 6.2 below proves the uniqueness of $\mu_{n_0,c}$ for n_0 sufficiently large: $\mu_{n_0,c}$ is a Lipshitz continuous function of $(\tilde{g}_{n_0}, R_{n_0}, \tilde{\mathbf{w}}_{n_0})$. In Theorem 6.3 below we prove that given μ_{n_0} as above there is a μ_0 given by a C^1 function of $\tilde{g}_0$ such that after n_0 applications of the RG map we arrive at μ_{n_0}.

To this end we represent the Banach space E_n as a product of two Banach spaces $E_n = E_{n,1} \times E_{n,2}$. We write $\upsilon_n \in E_n$ as $\upsilon_n = (\upsilon_{n,1}, \upsilon_{n,2})$ where $\upsilon_{n,1} = (\tilde{g}_n, R_n, \tilde{\mathbf{w}}_n)$ and $\upsilon_{n,2} = \mu_n$. $\upsilon_{n,2}$ is the expanding (relevant) variable. Let $p_i,\ i = 1, 2$, denote the projector onto $E_{n,i}$ and $f_{n,i} = p_i \circ f_n$. The norm $\|\cdot\|_n$ on E_n being a box norm we have

$$\|\upsilon_n\|_n = \max(\|\upsilon_{n,1}\|_n, \|\upsilon_{n,2}\|_n)$$

where $\|\upsilon_{n,2}\|_n = \bar{g}^{-(2-\delta)}|\mu_n|$ and $\|\upsilon_{n,1}\|_n = \max((\nu\bar{g})^{-1}|\tilde{g}_n|, \bar{g}^{-(11/4-\eta)}|||R_n|||_{\delta_n}, \tilde{c}_L^{-1}\|\tilde{\mathbf{w}}_n\|_{\delta_n})$.

In the following we continue to assume that L is sufficiently large, followed by ν sufficiently small depending on L, then ε sufficiently small depending on L. The last condition also implies that $\bar{g} \leq C_L\varepsilon$.

Theorem 6.2 *Let $\mathbf{s}_{n_0} = \{\upsilon_n : \upsilon_{n+1} = f_{n+1}(\upsilon_n)\}_{n\geq n_0} \in \mathbf{E}(1/4)$ be the global RG trajectory of Theorem 6.1. Then for n_0 sufficiently large there exists a Lipshitz continuous function $h : E_{n_0,1} \to \mathbb{R}$ with Lipshitz constant 1 such that the stable manifold of the sequence of maps $\{f_n\}_{n\geq n_0+1}$, $W^s_{n_0} = \{\upsilon_{n_0} \in E_{n_0}(1/32) : \mathbf{s}_{n_0} \in \mathbf{E}(1/4)\}$ is the graph*

$$W^s_{n_0} = \{\upsilon_{n_0,1}, h(\upsilon_{n_0,1}\}$$

We will prove the theorem following the analysis of Shub in [27, Sect. 5]. The Schub analysis has been employed earlier in the context of continuum models (see [7, Sect. 5.3] and [13, Sect. 6]). Here we have to take account of additional features stemming from the lattice which results in n_0 having to be taken sufficiently large (sufficiently fine lattice) for the argument to work. This will be clear from the proof of the following lemma from which Theorem 6.2 follows.

Lemma 6.3 *Let $\upsilon_n, \upsilon'_n \in E_n(1/4)$. Then for $n \geq n_0$, n_0 sufficiently large depending on ν and L*

$$\|f_{n+1,1}(\upsilon_n) - f_{n+1,1}(\upsilon'_n)\|_{n+1} \leq (1-\varepsilon)\|\upsilon_n - \upsilon'_n\|_n \tag{6.26}$$

and, if $\|\upsilon_{n,2} - \upsilon'_{n,2}\|_n \geq \|\upsilon_{n,1} - \upsilon'_{n,1}\|_n$ then

$$\|f_{n+1,2}(\upsilon_n) - f_{n+1,2}(\upsilon'_n)\|_{n+1} \geq (1+\varepsilon)\|\upsilon_n - \upsilon'_n\|_n \tag{6.27}$$

Proof First we prove (6.26). $f_{n+1,1}$ has components $f_{n+1,g}$, $f_{n+1,R}$ and $f_{n+1,w}$. From (6.3)

$$f_{n+1,g}(\upsilon_n) = \alpha(\epsilon)\tilde{g}_n + \tilde{\xi}_n(\upsilon_n)$$

Since $\upsilon_n, \upsilon_n' \in E_n(1/4)$ we can use Lemma 6.1. Therefore for $n \geq n_0$

$$\begin{aligned}
&(\nu\bar{g})^{-1}|f_{n+1,g}(\upsilon_n) - f_{n+1,g}(\upsilon_n')| \\
&\quad \leq \alpha(\epsilon)\|\upsilon_n - \upsilon_n'\|_n + (\nu\bar{g})^{-1}|\tilde{\xi}_n(\upsilon_n) - \tilde{\xi}_n(\upsilon_n')| \\
&\quad \leq \left(1 - \varepsilon\log(L) + C_L\varepsilon\left(\nu + \frac{1}{\nu}L^{-n_0 q} + \frac{1}{\nu}\varepsilon^{3/4-\eta}\right)\right)\|\upsilon_n - \upsilon_n'\|_n
\end{aligned}$$

Let L be large. Let ν be sufficiently small and n_0 *sufficiently large so that* $C_L L^{-n_0 q/2} \leq \nu \leq \frac{1}{C_L}$ and ε sufficiently small so that $\nu \geq C_L\varepsilon^{3/8-\eta}$. Then we have

$$\begin{aligned}
1 - \varepsilon\log(L) + \varepsilon C_L\left(\nu + \frac{1}{\nu}L^{-n_0 q} + \frac{1}{\nu}\varepsilon^{3/4-\eta}\right) &\leq 1 + \varepsilon(-\log(L) + 1 + L^{-qn_0/2} + \varepsilon^{3/8}) \\
&\leq 1 + \varepsilon(-\log(L) + 3) \leq 1 - \varepsilon
\end{aligned}$$

Therefore

$$(\nu\bar{g})^{-1}|f_{n+1,g}(\upsilon_n) - f_{n+1,g}(\upsilon_n')| \leq (1-\varepsilon)\|\upsilon_n - \upsilon_n'\|_n$$

Since $f_{n+1,R}(\upsilon_n) = U_{n+1}(\upsilon_n)$, we have from Lemma 6.1 for L sufficiently large

$$\bar{g}^{-(11/4-\eta)}|||f_{n+1,R}(\upsilon_n) - f_{n+1,R}(\upsilon_n')|||_{n+1} \leq (1-\varepsilon)\|\upsilon_n - \upsilon_n'\|_n$$

as well as

$$\tilde{c}_L^{-1}\|f_{n+1,\mathbf{w}}(\upsilon_n) - f_{n+1,\mathbf{w}}(\upsilon_n')\|_{n+1} \leq (1-\varepsilon)\|\upsilon_n - \upsilon_n'\|_n$$

These three inequalities prove (6.26).

Remark n_0 had to be chosen sufficiently large because the $\tilde{g}_n$ flow coefficient a_n, see (6.7), depends on the lattice scale n. a_n converges geometrically (Lemma 5.12) to a constant a_{c*}. As a result we have to wait sufficiently long before $\tilde{g}_n$ becomes irrelevant. A consequence of this is the presence of the $L^{-qn}\bar{g}^2$ term in the first inequality of Lemma 6.1. It has to be sufficiently small to ensure the validity of the first of three inequalities above.

Next we turn to (6.27). In this case by assumption $\|\upsilon_{n,2} - \upsilon_{n,2}'\| \geq \|\upsilon_{n,1} - \upsilon_{n,1}'\|$ and hence, since our norms are box norms, we have

$$\|\upsilon_n - \upsilon_n'\|_n = \|\upsilon_{n,2} - \upsilon_{n,2}'\|_n = \bar{g}^{-(2-\delta)}|\mu_n - \mu_n'|$$

From (6.4)

$$f_{n+1,\mu}(\upsilon_n) = L^{\frac{3+\varepsilon}{2}}\mu_n + \tilde{\rho}_n(\upsilon_n)$$

Then, using Lemma 6.1, we have

$$\begin{aligned}
\bar{g}^{-(2-\delta)}|f_\mu(\upsilon_n) - f_\mu(\upsilon_n')| &\geq L^{\frac{3+\varepsilon}{2}}\|\upsilon_n - \upsilon_n'\|_n - \bar{g}^{-(2-\delta)}|\tilde{\rho}(\upsilon_n) - \tilde{\rho}(\upsilon_n')| \\
&\geq (L^{\frac{3+\varepsilon}{2}} - C_L\bar{g}^{\delta})\|\upsilon_n - \upsilon_n'\|_n \\
&\geq (1+\varepsilon)\|\upsilon_n - \upsilon_n'\|_n
\end{aligned}$$

for ε sufficiently small depending on L. This proves (6.27). □

Proof of Theorem 6.2 Given Lemma 6.3, the proof follows the Schub argument as in [13]. Namely, to prove that $W^s_{n_0}$ is given by a graph of a function $\upsilon_{n_0,2} = h(y_{n_0,1})$ it is enough to prove that if in $W^s_{n_0}$ we take two points $\upsilon_{n_0} = (\upsilon_{n_0,1}, \upsilon_{n_0,2})$ and $\upsilon'_{n_0} = (\upsilon'_{n_0,1}, \upsilon'_{n_0,2})$ then

$$\|\upsilon_{n_0,2} - \upsilon'_{n_0,2}\|_{n_0} \le \|\upsilon_{n_0,1} - \upsilon'_{n_0,1}\|_{n_0} \tag{6.28}$$

because then for a given $\upsilon_{n_0,1}$ we would have at most one $\upsilon_{n_0,2}$, and by Theorem 6.1 there exists such a $\upsilon_{n_0,2}$. This means that $W^s_{n_0}$ is the graph of a function h, $\upsilon_{n_0,2} = h(\upsilon_{n_0,1})$, and moreover

$$\|h(\upsilon_{n_0,1}) - h(\upsilon'_{n_0,1})\|_{n_0} \le \|\upsilon_{n_0,1} - \upsilon'_{n_0,1}\|_{n_0}$$

Suppose (6.28) is not true. Then

$$\|\upsilon_{n_0,2} - \upsilon'_{n_0,2}\|_{n_0} > \|\upsilon_{n_0,1} - \upsilon'_{n_0,1}\|_{n_0} \tag{6.29}$$

Equation (6.29) implies that (6.27) holds. The latter followed by (6.26) gives

$$\begin{aligned}\|f_{n_0+1,2}(\upsilon_{n_0}) - f_{n_0+1,2}(\upsilon'_{n_0})\|_{n_0+1} &\ge (1+\varepsilon)\|\upsilon_{n_0} - \upsilon'_{n_0}\|_{n_0} > (1-\varepsilon)\|\upsilon_{n_0} - \upsilon'_{n_0}\|_{n_0} \\ &\ge \|f_{n_0+1,1}(\upsilon_{n_0}) - f_{n_0+1,1}(\upsilon'_{n_0})\|_{n_0+1}\end{aligned} \tag{6.30}$$

and hence

$$\begin{aligned}&\|f_{n_0+1}(\upsilon_{n_0}) - f_{n_0+1}(\upsilon'_{n_0})\|_{n_0+1} \\ &\quad = \|f_{n_0+1,2}(\upsilon_{n_0}) - f_{n_0+1,2}(\upsilon'_{n_0})\|_{n_0+1} \ge (1+\varepsilon)\|\upsilon_{n_0} - \upsilon'_{n_0}\|_{n_0}\end{aligned} \tag{6.31}$$

Define the composition of maps

$$\mathcal{P}^k_n \doteq f_{n+k} \circ \cdots \circ f_{n+2} \circ f_{n+1}$$

Now

$$\|\mathcal{P}^2_{n_0}(\upsilon_{n_0}) - \mathcal{P}^2_{n_0}(\upsilon'_{n_0})\|_{n_0+2} = \|f_{n_0+2}(f_{n_0+1}(\upsilon_{n_0})) - f_{n_0+2}(f_{n_0+1}(\upsilon'_{n_0}))\|_{n_0+2}$$

By (6.30) and the second part of Lemma 6.3 followed by (6.31) we get

$$\|\mathcal{P}^2_{n_0}(\upsilon_{n_0}) - \mathcal{P}^2_{n_0}(\upsilon'_{n_0})\|_{n_0+2} \ge (1+\varepsilon)\|f_{n_0+1}(\upsilon_{n_0}) - f_{n_0+1}(\upsilon'_{n_0})\|_{n_0+1} \ge (1+\varepsilon)^2\|\upsilon_{n_0} - \upsilon'_{n_0}\|_{n_0}$$

Repeating this k times we get for all $k \ge 0$

$$\|\mathcal{P}^k_{n_0}(\upsilon_{n_0}) - \mathcal{P}^k_{n_0}(\upsilon'_{n_0})\|_{n_0+k} \ge (1+\varepsilon)^k\|\upsilon_{n_0} - \upsilon'_{n_0}\|_{n_0} \tag{6.32}$$

Now $\upsilon_{n_0}, \upsilon'_{n_0}$ belong to $W^s_{n_0}$ and $\mathcal{P}^k_{n_0}(\upsilon_{n_0})$ is a member of the sequence $\mathbf{s}_{n_0} \in \mathbf{E}(1/4)$. Therefore $\|\mathcal{P}^k_{n_0}(\upsilon_{n_0})\|_{n_0+k} < 1/4$. Therefore we have from (6.32) the bound $\frac{1}{2} > (1+\varepsilon)^k\|\upsilon_{n_0} - \upsilon'_{n_0}\|_{n_0}$. By making k arbitrarily large we get a contradiction because $\upsilon_{n_0} \ne \upsilon'_{n_0}$ under (6.29). Hence (6.29) is true and the Theorem 6.2 has been proved. □

The next theorem establishes the uniqueness of the critical mass at the unit lattice scale.

Theorem 6.3 *Let n_0 and μ_{n_0} be as in Theorem 6.2. Let $\tilde{g}_0, \mu_0, R_0, \mathbf{w}_0$ belong to $\tilde{\mathcal{D}}_0$, defined in the beginning of Sect. 6.2. with $R_0 = 0, \mathbf{w}_0 = 0$. Let $U_1(1) = \{\tilde{g}_0 : 2^{(n_0+5)}(\nu\bar{g})^{-1}|\tilde{g}_0| < 1\}$. Let ε be sufficiently small depending on L and n_0. Then there exists an open ball $U_0 \subset U_1(1)$ and a C^1 function $h_0 : U_0 \to \mathbb{R}$ such that for $\mu_0 = h_0(\tilde{g}_0)$ the RG map applied n_0 times gives the effective critical mass μ_{n_0}.*

Remark This is the first time in our estimates that ε has been chosen to depend on n_0. Recall that in Lemma 6.3 n_0 was taken to be sufficiently large depending on ν and L.

Proof Let υ_n be as in (6.1). Let $r_n = 2^{-(n_0-n+5)}$ and $\lambda_n = L^{-\frac{(3+\varepsilon)}{2}(n_0-n)}$. Let $\tilde{E}_n$, $1 \le n \le n_0$, be the Banach space consisting of υ_n with norm

$$\|\upsilon_n\|_n = \max(r_n^{-1}(\nu\bar{g})^{-1}|\tilde{g}_n|,\ r_n^{-1}\lambda_n^{-1}\bar{g}^{-(2-\delta)}|\mu_n|,\ \bar{g}^{-(\frac{11}{4}-\eta)}|||R_n|||, \tilde{c}_L^{-1}\|\tilde{\mathbf{w}}_n\|_n) \tag{6.33}$$

We have $\tilde{E}_n \subset E_n$. $\tilde{E}_n(1)$ is the open unit ball in $\tilde{E}_n$. Note that $\tilde{E}_0(1)$ coincides with $\tilde{\mathcal{D}}_0$ as defined in the beginning of Sect. 6.2, for $R_0 = 0$ and $\mathbf{w}_0 = 0$, and $\tilde{E}_{n_0}(1) = E_{n_0}(\frac{1}{32})$. Then by Theorem 5.1 and Lemma 5.9 for $1 \le n \le n_0$ we have each RG map $f_n : \tilde{E}_n(1) \to \tilde{E}_{n+1}(1)$. Moreover each such map is (norm) analytic. Define the composition of maps

$$\mathcal{P}_0^{n_0} = f_{n_0} \circ f_{n_0-1} \circ \cdots \circ f_1 : \tilde{E}_0(1) \to \tilde{E}_{n_0}(1) \tag{6.34}$$

$\mathcal{P}_0^{n_0}$ is the composition of a finite number of analytic maps and therefore analytic. We consider the equation $\upsilon_{n_0} = \mathcal{P}_0^{n_0}(\upsilon_0)$ in the direction μ:

$$\mu_{n_0} = (\mathcal{P}_0^{n_0})_\mu(\upsilon_0) \tag{6.35}$$

with $\upsilon_0 = (\tilde{g}_0, \mu_0, 0, \tilde{\mathbf{w}}_0)$ with $\mathbf{w}_0 = 0$ (recall that $\tilde{\mathbf{w}}_0 = \mathbf{w}_0 - \mathbf{w}_*$).

We will solve (6.33) for μ_0 for fixed μ_{n_0} using the (Banach space) implicit function theorem. Let $x = (\tilde{g}_0, \mu_{n_0})$ and $y = \mu_0$. We have set $\mathbf{w}_0 = R_0 = 0$. Let V_1 be the Banach space of elements x with norm

$$\|x\| = \max(r_0^{-1}(\nu\bar{g})^{-1}|\tilde{g}_0|,\ r_{n_0}^{-1}\bar{g}^{-(2-\delta)}|\mu_{n_0}|)$$

Let V_2 be the Banach space of elements y with norm

$$\|y\| = r_0^{-1}\lambda_0^{-1}\bar{g}^{-(2-\delta)}|\mu_0|$$

Let $V_j(r)$ be the open ball in V_j of radius r, centered at the origin. Define

$$F(x, y) = (\mathcal{P}_0^{n_0})_\mu(\upsilon_0) - \mu_{n_0} \tag{6.36}$$

Solving (6.35) is equivalent to solving $F(x, y) = 0$ for y.

Recall that $\upsilon_0 = (\tilde{g}_0, \mu_0, 0, \tilde{\mathbf{w}}_0)$. We have $F(0, 0) = 0$ and $F(\cdot, \cdot) : V_1(1) \times V_2(1) \to \mathbb{R}$ is an analytic map and therefore C^2. Taking a y derivative of $F(x, y)$ gives $D_y F(x, y) = D_{\mu_0}(\mathcal{P}_0^{n_0})_\mu(\upsilon_0)$. We will prove that the linear map

$$D_y F(0, 0) : V_2 \to \mathbb{R}$$

is injective. It is easy to see that

$$(\mathcal{P}_0^{n_0})_\mu(\upsilon_0) = L^{\frac{(3+\varepsilon)}{2}n_0}\left(\mu_0 + L^{-\frac{(3+\varepsilon)}{2}}\sum_{j=0}^{n_0-1} L^{-\frac{(3+\varepsilon)}{2}j}\tilde{\rho}_j(\upsilon_j)\right)$$
$$\tilde{\rho}_j(\upsilon_j) = \tilde{\rho}_j \circ (\mathcal{P}_0^j)_\mu(\upsilon_0) \tag{6.37}$$

$\tilde{\rho}_j$ was defined earlier in (6.9) and is analytic in $E_j(1)$. The map $\tilde{\rho}_j \circ (\mathcal{P}_0^j)_\mu$ is analytic since it is a composition of analytic maps. Let $\mu_0 \in V_2(\frac{1}{4})$. Let γ be the closed contour $\gamma = \{\mu : \mu - \mu_0 = Re^{i\theta}\}$ with $R = \frac{1}{4}r_0\lambda_0\bar{g}^{(2-\delta)}$. We estimate the μ derivative of $\tilde{\rho}_j \circ (\mathcal{P}_0^j)_\mu(\upsilon_0)$ by using the Cauchy integral formula integrating along the contour γ enclosing a pole at μ_0 together with the estimate for $\tilde{\rho}_j(\upsilon_j)$ given in (6.13) which is valid in $E_j(1)$. The latter is guaranteed by our choice of contour. We have

$$\left|D_{\mu_0}\tilde{\rho}_j \circ (\mathcal{P}_0^j)_\mu(\upsilon_0)\right|_{\tilde{g}_0=\mu_0=0} \le \frac{c_L\bar{g}^2}{R} \le c_{L,n_0}\bar{g}^\delta$$

Using this estimate we get from (6.37)

$$D_{\mu_0}(\mathcal{P}_0^{n_0})_\mu(\upsilon_0)\Big|_{\tilde{g}_0=\mu_0=0} \ge L^{\frac{(3+\varepsilon)}{2}n_0}(1 - c'_{L,n_0}\bar{g}^\delta)$$

Taking ε sufficiently small depending on L and n_0 makes $\bar{g}$ sufficiently small so as to ensure $D_{\mu_0}(\mathcal{P}_0^{n_0})_\mu(\upsilon_0)|_{\tilde{g}_0=\mu_0=0} \ge \frac{1}{2}$. Therefore the map $D_yF(0,0) : V_2 \to \mathbb{R}$ is injective. Hence by the implicit function theorem there exists a ball $\tilde{U}_0$ containing $x = 0$ with $\tilde{U}_0 \subset V_1(1)$, and a C^1 function $\tilde{h}_0$ in $\tilde{U}_0$ with $\tilde{h}_0(x) \in V_2(\frac{1}{2})$ such that $F(x, \tilde{h}_0(x)) = 0$. For ε sufficiently small depending on L and n_0 we have $\mu_{n_0} \in \tilde{U}_0$. This completes the proof of the theorem because for $\mu_{n_0} \in \tilde{U}_0$, $\tilde{U}_0$ restricts to the ball U_0 and correspondingly $\tilde{h}_0$ restricts to the desired function h_0. □

Theorems 6.2 and 6.3 put together completes our construction of the stable manifold starting from the unit lattice.

Finally we remark that as a consequence of Theorem 6.2 we have $\upsilon_n \in E_n(1/4)$, $\forall n \ge n_0$. This implies that $|\tilde{g}_n| < \frac{1}{4}\nu\bar{g}$, $\forall n \ge n_0$. By construction the same statement is also true for $0 \le n \le n_0$. Whence for all $n \ge 0$

$$\left(1 - \frac{1}{4}\nu\right)\bar{g} < g_n < \left(1 + \frac{1}{4}\nu\right)\bar{g} \tag{6.38}$$

We have $0 < \nu < \frac{1}{2}$. Therefore the effective coupling constant generated by the discrete RG flow is uniformly bounded away from 0 at all RG scales.

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